

# OPERATIONAL GALOIS ADJUNCTIONS<sup>1</sup>

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We present a detailed synthetic overview of the utilisation of categorical techniques in the study of order structures together with their applications in operational quantum theory. First, after reviewing the notion of residuation and its implementation at the level of quantaloids we consider some standard universal constructions and the extension of adjunctions to weak morphisms. Second, we present the categorical formulation of closure operators and introduce a hierarchy of contextual enrichments of the quantaloid of complete join lattices. Third, we briefly survey physical state-property duality and the categorical analysis of derived notions such as causal assignment and the propagation of properties.

## 1. Introduction

The starting point for the structure theory we shall expose in this paper is the well known fact that preordered sets may be considered as small thin categories; one can then not only reformulate a large part of the theory of order structures in categorical terms, but also apply general categorical techniques to specific order

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theoretic problems. We provide a brief introduction to category theory in section 2; for detailed presentations see for example [Adámek, Herrlich and Strecker 1990; Borceux 1994; Mac Lane 1971]. First, as discussed in section 3, the notion of an adjunction reduces to that of a residuation; the resulting coisomorphy between the categories of join complete lattices and meet complete lattices will provide a guiding principle for the rest of this work. For general expositions of residuation theory see [Blyth and Janowitz 1972; Derdérian 1967]. Second, as discussed in section 4, the consideration of simple examples allows the direct characterisation of special morphisms and thereby both the construction of limits and the definition of pseudoadjoints for weak morphisms. Third, as discussed in section 5, following [Moore 1995, 1997, 2000] the definition of a monad reduces to that of a closure operator. In particular, the categories of atomistic join complete lattices and closure spaces are equivalent; for a general discussion of the categorical algebra of matroids see [Faure 1994; Faure and Frölicher 1996, 1998]. Fourth, as discussed in section 6, following [Amira, Coecke and Stubbe 1998; Coecke and Stubbe 1999a,b, 2000] the passage from the static consideration of individual lattices to the dynamic viewpoint of induced quantaloids allows the introduction of an inclusion hierarchy of structures representing successive levels of contextual enrichment. For general expositions of quantaloid theory see [Rosenthal 1991]. Fifth, far from being of purely technical interest, this categorical formalism has direct application in operational quantum theory, as developed in [Aerts 1982, 1994; Jauch and Piron 1969; Piron 1964, 1976, 1990]. In particular, as discussed in section 7, following [Moore 1999] the categorical equivalence between orthogonal spaces and atomistic complete ortholattices determined by the existence of monadic comparison functors has a direct interpretation in terms of the primitive duality between the state and property descriptions of a physical system. By way of application, following [Faure and Frölicher 1993, 1994, 1995] one can then reformulate the Hilbertian representation of projective orthogeometries in purely categorical terms. Note that our exposition focuses on the basic structure theory of adjunctions on complete lattices. As such we shall not discuss topics such as orthoadjunctions on orthomodular lattices, an important subject leading to the analysis of Baer  $*$ -semigroups via the Sasaki projection [Foulis 1960, 1962] and the action of conditioning maps on weight spaces [Foulis and Randall 1971, 1974; Frazer, Foulis and Randall 1980]. For extensions to test spaces see [Bennett and Foulis 1998; Wilce 2000], and for a logical analysis of perfect measurements in this context see [Coecke and Smets 2000].

For readability and ease of presentation we have relegated proofs to an appendix. Note that, while several of our results are either new or genuine extensions of standard ones, we would like to emphasise our uniform presentation of the theory and not just its novelty. As such, we have made no attempt to trace the historical origins and development of our categorical approach, preferring to emphasise its coherence as a synthetic tool and its utility in applications. Similarly, our bibliography should be taken as indicative rather than exhaustive. For the reader's convenience, we end this introduction by collecting together the definitions of the main categories to be treated in the following. We will use abbreviations when denoting Hom-sets, e.g.,  $J(L_1, L_2)$  for those of JCLatt. First, any join preserving

map  $f : L_1 \rightarrow L_2$  satisfies the condition  $f(0_1) = f(\bigvee_1 \emptyset) = \bigvee_2 f(\emptyset) = \bigvee_2 \emptyset = 0_2$ . We then obtain the following hierarchy of categories of complete lattices :

Category	Map preservation	Constraints
WJCLatt	Non-empty joins	
JCLatt	Arbitrary joins	Note: $f(0_1) = 0_2$
BJCLatt	Arbitrary joins, balanced	$f(1_1) = 1_2$
DJCLatt	Arbitrary joins, dense	$f(a_1) = 0_2 \Rightarrow a_1 = 0_1$

Dualising we obtain the analogous categories of meet preserving maps. Second, the Galois adjunction provides an isomorphism between JCLatt and MCLatt<sup>op</sup>, which restricts to isomorphisms BJCLatt  $\simeq$  DMCLatt<sup>op</sup> and DJCLatt  $\simeq$  BMCLatt<sup>op</sup>. We shall show that this isomorphism can be extended to weak morphisms in two equivalent manners. Explicitly, for  $L$  a complete lattice let  $[0, a] = \{x \in L \mid x < a\}$  and  $L^u = L \dot{\cup} \{1\}$ . Then WMCLatt<sup>op</sup> is isomorphic to each of the categories :

Category	Hom-sets	Morphisms
PJCLatt	$PJ(L_1, L_2) = \bigcup_{a \in L_1} J([0, a_1], L_2)$	Sectional maps
UJCLatt	$UJ(L_1, L_2) = BJ(L_1^u, L_2^u)$	Upper maps

Third, for complete atomistic lattices the duality between JCLatt and MCLatt also restricts to atomic morphisms. Explicitly, let  $\Sigma_L$  be the set of atoms of  $L$  and  $\alpha : \Sigma_1 \setminus K_1 \rightarrow \Sigma_2$  be a continuous partial map between the closure spaces  $(\Sigma_1, T_1)$  and  $(\Sigma_2, T_2)$ . We then obtain the dual categories :

Category	Morphisms	Induced from closure
JCALatt	$f(\Sigma_{L_1}) \subseteq \Sigma_{L_2}$	$f_\alpha : A_1 \mapsto T_2 f(A_1 \setminus K_1)$
MCALatt	$(\forall p_1 \exists p_2) p_1 < g(p_2)$	$g_\alpha : A_2 \mapsto K_1 \cup f^{-1}(A_2)$

Note that the categories JCALatt and CSpace are equivalent. Fourth, applying the power construction to complete lattices we obtain an inclusion hierarchy of quantaloids expressing successive degrees of contextual enrichment. Explicitly, define  $P_0(L) = P(L \setminus \{0\})$ , and for  $f \in J(L_1, L_2)$  and  $\theta \in J(P_0(L_1), P_0(L_2))$  let us write  $f \succ \theta$  if  $f(\bigvee A_1) = \bigvee \theta(A_1)$  for each  $A_1 \in P_0(L_1)$ . We then obtain :

Category	Morphisms	Name
PStruct	$P_f$ for $f \in J(L_1, L_2)$	Power structures
BStruct	$\bigcup_\alpha P_{f_\alpha}$ for $f_\alpha \in J(L_1, L_2)$	Based structures
TStruct	$(f, \theta)$ for $f \succ \theta$	Transition structures
FStruct	$\theta \in J(P_0(L_1), P_0(L_2))$	Functional structures

## 2. Category theory

At the most naive level, category theory may be construed as a hierarchy of object-structure relations, the standard definitions then reducing to unicity requirements for induced relations. First, if the morphism  $f$  relates the objects  $A$  and  $B$  and the morphism  $g$  relates the objects  $B$  and  $C$  then it is natural to suppose the existence of an induced relation  $g \circ f$  between  $A$  and  $C$ : since identification provides a canonical relation between  $A$  and itself we are then led to suppose the existence of identity morphisms as compositional units; since  $h \circ (g \circ f)$  and  $(h \circ g) \circ f$  are both induced relations between  $A$  and  $D$  we are led to require associativity considered as order indifference of concatenation. We then recover the usual definition of a category. Second, a relation between morphisms should respect the structurally important features involved in the concept of a morphism: a functor  $F$  should then relate the domain and codomain objects of the initial morphism to those of the final morphism, so that  $Ff : FA \rightarrow FB$  for  $f : A \rightarrow B$ : identity morphisms form a distinguished class and so should be preserved; the two induced relations  $F(g \circ f)$  and  $Fg \circ Ff$  between  $FA$  and  $FB$  should coincide. We then recover the usual definition of a functor. Third, any relation  $\theta$  between two functors  $F$  and  $G$  should induce a relation or morphism  $\theta_A$  between  $FA$  and  $GA$ : further for  $f : A \rightarrow B$  the two induced relations  $\theta_B \circ Ff$  and  $Gf \circ \theta_A$  between  $FA$  and  $GB$  should coincide. We then recover the usual definition of a natural transformation.

Much of the power of category theory is due to the notion of universal constructions. A local approach is through the dual notions of limits and colimits of diagrams, where a diagram is an indexed set of objects subject to structural relations, that is, a functor  $\nabla : \underline{\mathbf{J}} \rightarrow \underline{\mathbf{X}}$  where  $\underline{\mathbf{J}}$  is a small index category encoding the constraints. Now, to relate diagrams to compatible objects it suffices to remark that each object  $A$  can be modeled by the corresponding constant functor  $C_A : \underline{\mathbf{J}} \rightarrow \underline{\mathbf{X}}$ . By definition, a source is then a natural transformation  $p : C_A \rightarrow \nabla$ . In this way we obtain limits as distinguished sources  $p$  such that for any other source  $\overline{p}$  there exists a unique morphism  $f$  satisfying  $\overline{p} = p \circ f$ . On the other hand, a global approach to universal constructions is provided by the notion of adjoint functors. Explicitly, let  $F : \underline{\mathbf{X}} \rightarrow \underline{\mathbf{Y}}$  and  $G : \underline{\mathbf{Y}} \rightarrow \underline{\mathbf{X}}$ . Now we can only directly relate functors with the same domain and codomain. We are then led to define  $F \dashv G$  if there exist natural transformations  $\eta : \text{Id}_{\underline{\mathbf{X}}} \rightarrow G \circ F$  and  $\varepsilon : F \circ G \rightarrow \text{Id}_{\underline{\mathbf{Y}}}$  satisfying the coherence conditions  $\varepsilon F \circ F\eta = \text{id}_F$  and  $G\varepsilon \circ \eta G = \text{id}_G$ . Note that the two conceptions of universality are closely interrelated. For example, if  $F \dashv G$  then  $G$  preserves limits and  $F$  preserves colimits, and any two (co)adjoints of the same functor are naturally isomorphic. Further, products, defined as limits of trivial diagrams whose index category has only identity morphisms, may be globalised as adjoints of the appropriate diagonal functors. Finally, two categories are called equivalent if there exist natural isomorphisms  $\varphi : \text{Id}_{\underline{\mathbf{X}}} \rightarrow G \circ F$  and  $\psi : \text{Id}_{\underline{\mathbf{Y}}} \rightarrow F \circ G$  satisfying the equivalent coherence conditions  $F\varphi = \psi F$  and  $\varphi G = G\psi$ . Note that in this case  $F \dashv G \dashv F$ , the two functors  $F$  and  $G$  then preserving (co)limits and adjunctions.

Often one is interested in categories  $\underline{A}$  which can be considered as specialisations of some base category  $\underline{X}$ . For example, we may wish to treat objects in  $\underline{A}$  as objects in  $\underline{X}$  together with extra structure, and morphisms in  $\underline{A}$  as morphisms in  $\underline{X}$  which respect that structure in some sense. We are then led to define a concrete category over the category  $\underline{X}$  to be a pair  $(\underline{A}, U)$ , where  $\underline{A}$  is a category and  $U$  is a faithful functor from  $\underline{A}$  to  $\underline{X}$ , that is, a functor which is injective on Hom-sets. It is then of interest to consider those morphisms  $f : UA \rightarrow UB$  in  $\underline{X}$  which lift, in the sense that there exists  $\varphi : A \rightarrow B$  with  $U\varphi = f$ . For example,  $\varphi : A \rightarrow B$  is called initial if  $f : UA' \rightarrow UA$  lifts whenever  $U\varphi \circ f$  does, or final if  $g : UB \rightarrow UB'$  lifts whenever  $f \circ U\varphi$  does. Next, each fibre  $F(X) = U^{-1}(X)$  has a natural preorder defined by  $A < B$  if  $\text{id}_X : UA \rightarrow UB$  lifts. In particular, categories whose fibres are ordered by equality have algebraic character, whereas those whose fibres are complete lattices have topological character. Finally, an important step in category theory is to replace Hom-sets by objects in an appropriate structure category, leading to the notion of enrichment. Explicitly, in order to define composition adequately we require that the structure category  $\underline{V}$  be symmetric monoidal closed, in the sense that there exists a tensor functor  $\otimes : \underline{V} \times \underline{V} \rightarrow \underline{V}$  and unit object  $I \in \text{Ob}(\underline{V})$  such that the usual composition laws may be replaced by coherent natural transformations. For example, standard categories are just categories over  $\underline{\text{Set}}$ , 2-categories are defined to be categories over  $\underline{\text{Cat}}$ , and quantaloids may be considered as categories enriched in join complete lattices [Borceux and Stubbe 2000]. Quantales then represent the monoidal one-object restrictions [Paseka and Rosický 2000]. We obtain quantale and quantaloid morphisms as the corresponding  $\underline{\text{JCLatt}}$ -enriched functors.

Finally, much of the theory of order structures can be reformulated in purely categorical terms by remarking that preordered classes are in bijective correspondence with thin categories, namely categories for which each Hom-set has at most one element. Explicitly,  $a < b$  if and only if  $\text{Hom}(a, b) \neq \emptyset$ , reflexivity being the existence of identity morphisms and transitivity being the condition of morphism composability. Note that the unicity of a morphism  $\alpha : a \rightarrow b$  implies that any diagram which can be written must commute. In particular, products are exactly meets,  $(\forall \alpha \in \Omega) (x < a_\alpha) \Leftrightarrow x < \bigwedge_\alpha a_\alpha$ , whereas coproducts are exactly joins,  $(\forall \alpha \in \Omega) (a_\alpha < x) \Leftrightarrow \bigvee_\alpha a_\alpha < x$ . Further, a functor between two preordered classes is exactly an isotone map, preservation of order being exactly preservation of composition. In particular, there exists a natural transformation  $\theta : f \rightarrow g$  if and only if  $f < g$ , this being the necessary and sufficient condition for the existence of morphisms  $\theta_a : f(a) \rightarrow g(a)$ . For isotone maps  $f : L_1 \rightarrow L_2$  and  $g : L_2 \rightarrow L_1$ , we then have that  $f \dashv g$  if and only if  $\text{id}_1 < (g \circ f)$  and  $(f \circ g) < \text{id}_2$ , these being the necessary and sufficient conditions for the existence of natural transformations  $\eta : \text{id}_1 \rightarrow (g \circ f)$  and  $\varepsilon : (f \circ g) \rightarrow \text{id}_2$ . Note that in this context quantaloids are exactly locally complete and thin 2-categories. For convenience, in the following we shall restrict our attention to posets, that is, small thin categories for which no two distinct elements are isomorphic. For such categories natural isomorphy reduces to identity, so that adjoints are unique when they exist.

### 3. Morphisms and adjunctions

In this preliminary section we recall some elementary facts about adjunctions on posets considered as thin categories, results which form the core of the rest of this work. Explicitly, we start by transcribing the definition and basic properties of adjunctions in the context of posets, before turning to limit preservation properties in the context of complete lattices. We then globalise these observations to categories of posets and finish with some remarks on the orthocomplemented case. Let  $f$  and  $g$  be isotone maps on posets. Then :

- [3.1.1]  $\text{id}_1 < (g \circ f) \ \& \ (f \circ g) < \text{id}_2$  iff  $f(a_1) < a_2 \Leftrightarrow a_1 < g(a_2)$ ;
- [3.1.2]  $f$  is an isomorphism with inverse  $g$  iff  $f \dashv g \dashv f$ ;
- [3.1.3] If  $f \dashv g$  then  $f \circ g \circ f = f$  and  $g \circ f \circ g = g$ ;
- [3.1.4] If  $f \dashv g$  and  $\bar{f} \dashv \bar{g}$  then  $f < \bar{f} \Leftrightarrow \bar{g} < g$ ;
- [3.1.5]  $\text{id} \dashv \text{id}$ , and if  $f \dashv g$  and  $\bar{f} \dashv \bar{g}$  then  $(\bar{f} \circ f) \dashv (g \circ \bar{g})$ .

The first result gives a more practical form for the adjunction condition, whereas the second encodes isomorphy as equivalence. The third result establishes adjunctions as pseudoinverses. As we shall see later, the fourth result enables a globalisation of adjunctions from the level of individual posets to the level of categories of posets, the fifth leading to a natural generalisation to quantaloids. Next, transcribing the limit preservation properties of adjunctions we obtain the following results for isotone maps on complete lattices :

- [3.2.1] If  $f \dashv g$  then  $f(\bigvee A_1) = \bigvee f(A_1)$  and  $g(\bigwedge A_2) = \bigwedge g(A_2)$ ;
- [3.2.2] If  $f(\bigvee A_1) = \bigvee f(A_1)$  then  $f \dashv f^*: L_2 \rightarrow L_1 : a_2 \mapsto \bigvee \{a_1 \in L_1 \mid f(a_1) < a_2\}$ ;
- [3.2.3] If  $g(\bigwedge A_2) = \bigwedge g(A_2)$  then  $g \vdash g_*: L_1 \rightarrow L_2 : a_1 \mapsto \bigwedge \{a_2 \in L_2 \mid a_1 < g(a_2)\}$ ;
- [3.2.4] If  $f_\alpha \dashv g_\alpha$  then  $\bigvee_\alpha f_\alpha \dashv \bigwedge_\alpha g_\alpha$ ;
- [3.2.5] If  $f \dashv g$ ,  $f_\alpha \dashv g_\alpha$  then  $f \circ (\bigvee_\alpha f_\alpha) \dashv \bigwedge_\alpha (g_\alpha \circ g)$ ,  $(\bigvee_\alpha f_\alpha) \circ f \dashv \bigwedge_\alpha (g \circ g_\alpha)$ .

The first result implies that join (meet) preservation is a necessary condition for the existence of a right (left) adjoints, the second and third implying sufficiency. The fourth result implies that the set  $J(L_1, L_2)$  of join preserving maps between the complete lattices  $L_1$  and  $L_2$  is a complete lattice with respect to the pointwise join, whereas the set  $M(L_2, L_1)$  of meet preserving maps between the complete lattices  $L_2$  and  $L_1$  is a complete lattice with respect to the pointwise meet. Finally, the fifth result implies that composition distributes on both sides over joins in JCLatt and meets in MCLatt. In particular, the category JCLatt provides the paradigm example of a quantaloid.

In our last remarks we have implicitly used Birkhoff's theorem, which states that any join complete lattice is also meet complete and conversely. Note that this result may be construed as an application of the adjoint functor theorem to the indexed diagonal functor  $\Delta : L \rightarrow \times_\alpha L : a \mapsto (a_\alpha = a)$ , since  $L$  is complete if and

only if  $J \dashv \Delta \dashv M$ , with  $J$  the join and  $M$  the meet. Nevertheless, as is typical in category theory, join and meet completeness are rather different at the level of morphisms. For example, let  $M(L_1, L_2)^{\text{co}}$  be the complete lattice of meet preserving maps with opposite pointwise order,  $\underline{\text{MCLatt}}^{\text{op}}$  be the category of complete lattices with meet preserving maps and opposite Hom-sets, and  $\underline{\text{MCLatt}}^{\text{coop}}$  be the quantaloid of complete lattices with meet preserving maps, opposite pointwise order and opposite Hom-sets. Considering the families of maps  $A^*: L \mapsto L$ ;  $f \mapsto f^*$  and  $A_*: L \mapsto L$ ;  $g \mapsto g_*$  we then obtain :

[3.3.1]  $J(L_1, L_2)$  is isomorphic as a complete lattice to  $M(L_2, L_1)^{\text{co}}$ ;

[3.3.2]  $\underline{\text{JCLatt}}$  is isomorphic as a category to  $\underline{\text{MCLatt}}^{\text{op}}$ ;

[3.3.3]  $\underline{\text{JCLatt}}$  is isomorphic as a quantaloid to  $\underline{\text{MCLatt}}^{\text{coop}}$ .

Finally, the above dualities can be usefully restricted to orthocomplemented lattices, that is, bounded lattices  $L$  equipped with an operation  $' : L \rightarrow L$  satisfying :  $a < b \Rightarrow b' < a'$ ;  $a'' = a$ ;  $a \wedge a' = 0$ . Explicitly, for  $\alpha : L_1 \rightarrow L_2$  a map between orthocomplemented lattices let  $C(\alpha) : L_1 \rightarrow L_2 : a_1 \mapsto \alpha(a_1)'$  be the conjugate map. For  $f \dashv g$  we then define the orthoadjoints  $f^\dagger : L_2 \rightarrow L_1 : a_2 \mapsto g(a_2)'$  and  $g^\dagger : L_1 \rightarrow L_2 : a_1 \mapsto f(a_1)'$ . Let us write  $\underline{\text{IoLatt}}$  for the category of orthocomplemented lattices with isotone maps,  $\underline{\text{JCoLatt}}$  and  $\underline{\text{MCoLatt}}$  for the categories of complete orthocomplemented lattices with respectively join or meet preserving maps, and  $\underline{\text{COLatt}}$  for the category of complete lattices with maps preserving the join, the meet, and the orthocomplementation. Then :

[3.4.1]  $C$  is an endofunctor on  $\underline{\text{IoLatt}}$  restricting to  $\underline{\text{JCoLatt}} \simeq \underline{\text{MCoLatt}}$ ;

[3.4.2] In  $\underline{\text{JCoLatt}}$  we have  $(f_1 \circ f_2)^\dagger = f_2^\dagger \circ f_1^\dagger$ ,  $f^{\dagger\dagger} = f$ ,  $f^\dagger \circ f = 0_1 \Leftrightarrow f = 0_2$ ;

[3.4.3] In  $\underline{\text{JCoLatt}}$  we have  $u^\dagger \circ u = \text{id}$  iff  $a_1 < b_1' \Leftrightarrow u(a_1) < u(b_1)'$ ;

[3.4.4] In  $\underline{\text{COLatt}}$  we have  $h_*(a_2') = h^*(a_2)'$ ,  $h^\dagger = h_*$ ,  $h \circ h^\dagger \circ h = h$ .

The first result implies that  $\underline{\text{JCoLatt}}$  is self-dual. The second result exhibits  $\underline{\text{JCoLatt}}$  as a regular  $\dagger$ -semigroup. The third result classifies isometries in  $\underline{\text{JCoLatt}}$ , whereas the fourth implies that all orthomorphisms are partially isometric.

#### 4. Special morphisms

As is usual in category theory, it is important to have a number of examples of particular morphisms such as constant maps or subobject inclusions. First, for  $\mathbf{2}$  the two-element lattice and  $[0, a] = \{x \in L \mid x < a\}$  a lower interval, let

$$\begin{aligned} \alpha_a : \mathbf{2} \rightarrow L : 0 \mapsto 0; 1 \mapsto a & \quad C^a : L \rightarrow \mathbf{2} : x \mapsto 1 (a < x); 0 (a \not< x); \\ \alpha^a : \mathbf{2} \rightarrow L : 0 \mapsto a; 1 \mapsto 1 & \quad C_a : L \rightarrow \mathbf{2} : x \mapsto 0 (x < a); 1 (x \not< a); \end{aligned}$$

$$\begin{aligned} i_a : [0, a] \rightarrow L : x \mapsto x & \quad \hat{i}_a : [0, a] \rightarrow L : x \mapsto x (x \neq a); 1 (x = a); \\ \pi_a : L \rightarrow [0, a] : x \mapsto x \wedge a & \quad \hat{\pi}_a : L \rightarrow [0, a] : x \mapsto x (x < a); a (x \not< a). \end{aligned}$$

For any adjunction  $f \dashv g$  we then obtain :

$$[4.1.1] \quad \alpha_a \dashv C^a \text{ with } f \circ \alpha_{a_1} = \alpha_{f(a_1)} \text{ and } C^{a_1} \circ g = C^{f(a_1)} ;$$

$$[4.1.2] \quad C_a \dashv \alpha^a \text{ with } g \circ \alpha^{a_2} = \alpha^{g(a_2)} \text{ and } C_{a_2} \circ f = C_{g(a_2)} ;$$

$$[4.1.3] \quad i_a \dashv \pi_a \text{ with } \pi_a \circ i_a = \text{id}, \text{ and } \hat{\pi}_a \dashv \hat{i}_a \text{ with } \hat{\pi}_a \circ \hat{i}_a = \text{id}.$$

Note that the  $\alpha_a$  exhaust join preserving maps  $\alpha : \mathbf{2} \rightarrow L$ , since any such map must satisfy  $\alpha(0) = 0$ , whereas the  $\alpha^a$  exhaust meet preserving maps  $\alpha : \mathbf{2} \rightarrow L$ , since any such map must satisfy  $\alpha(1) = 1$ . In particular, the  $C^a$  exhaust meet preserving maps  $C : L \rightarrow \mathbf{2}$  whereas the  $C_a$  exhaust join preserving maps  $C : L \rightarrow \mathbf{2}$ . Here the restriction to complete join maps is essential; indeed the kernels of finite join maps  $f : L \rightarrow \mathbf{2}$  are exactly the ideals on  $L$ , such an ideal being prime if and only if  $f$  also preserves finite meets. Now, if  $f : L_1 \rightarrow L_2$  preserves joins then  $f(0_1) = 0_2$ : we then call  $f$  balanced if  $f(1_1) = 1_2$  or dense if  $f(a_1) = 0_2 \Leftrightarrow a_1 = 0_1$ . Dually, if  $g : L_2 \rightarrow L_1$  preserves meets then  $g(1_2) = 1_1$ : we then call  $g$  balanced if  $g(0_2) = 0_1$  or dense if  $g(a_2) = 1_1 \Leftrightarrow a_2 = 1_2$ . Note that a morphism in either JCLatt or MCLatt is injective only if it is dense, or surjective only if it is balanced. Recall that  $h$  is called an epimorphism if  $h_1 \circ h = h_2 \circ h \Rightarrow h_1 = h_2$ , or a monomorphism if  $h \circ h_1 = h \circ h_2 \Rightarrow h_1 = h_2$ . For  $f \dashv g$  an adjunction we then obtain :

$$[4.2.1] \quad f \text{ is balanced iff } g \text{ is dense, or dense iff } g \text{ is balanced ;}$$

$$[4.2.2] \quad f \text{ is epic iff } f \text{ is surjective iff } f \circ g = \text{id}_2 \text{ iff } g \text{ is injective iff } g \text{ is monic ;}$$

$$[4.2.3] \quad f \text{ is monic iff } f \text{ is injective iff } g \circ f = \text{id}_1 \text{ iff } g \text{ is surjective iff } g \text{ is epic .}$$

The first result implies that the categories BJCLatt with balanced join morphisms and DMCLatt<sup>op</sup> with opposite dense meet morphisms are isomorphic, as are the categories DJCLatt and BMCLatt<sup>op</sup>. The second and third results render explicit the general duality between epimorphisms and monomorphisms together with their standard set theoretic interpretations. Finally, if  $\bar{f} \circ f = \text{id}$  then  $f$  is called a section and  $\bar{f}$  is called a retraction; each section is monic and each retraction is epic. For example,  $(\hat{\pi}_a \circ i_a)(x) = \hat{\pi}_a(x) = x$ , so that  $\hat{\pi}_a$  is a retraction and  $i_a$  is a section in JCLatt. Similarly,  $(\pi_a \circ \hat{i}_a)(x) = [\pi_a(x) (x \neq a); \pi_a(1) (x = a)] = x$ , so that  $\pi_a$  is a retraction and  $\hat{i}_a$  is a section in MCLatt.

Next, recall that the direct product  $\times_\alpha L_\alpha$  of the family of bounded posets  $L_\alpha$  is the Cartesian product of the  $L_\alpha$  equipped with the pointwise order. We may then define the maps :

$$\Pi_\beta : \times_\alpha L_\alpha \rightarrow L_\beta : (a_\alpha) \mapsto a_\beta ;$$

$$i_\beta : L_\beta \rightarrow \times_\alpha L_\alpha : b \mapsto (a_\alpha) \text{ with } a_\alpha = b \text{ } (\alpha = \beta) ; 0 \text{ } (\alpha \neq \beta) ;$$

$$j_\beta : L_\beta \rightarrow \times_\alpha L_\alpha : b \mapsto (a_\alpha) \text{ with } a_\alpha = b \text{ } (\alpha = \beta) ; 1 \text{ } (\alpha \neq \beta) .$$

On the other hand, recall that the horizontal sum  $\dot{\cup}_\alpha L_\alpha$  of the  $L_\alpha$  is the disjoint union of the  $L_\alpha \setminus \{0_\alpha, 1_\alpha\}$  with componentwise order and adjoined minimal and maximal elements. We may then define the maps :



$$\begin{aligned} I_\beta &: L_\beta \rightarrow \dot{\cup}_\alpha L_\alpha : b \mapsto b; \\ \sigma_\beta &: \dot{\cup}_\alpha L_\alpha \rightarrow L_\beta : x \mapsto x \ (x \in L_\beta); \ 1 \ (x \notin L_\beta); \\ \rho_\beta &: \dot{\cup}_\alpha L_\alpha \rightarrow L_\beta : x \mapsto x \ (x \in L_\beta); \ 0 \ (x \notin L_\beta). \end{aligned}$$

Finally, recall that a product of the family of objects  $L_\beta$  is an object  $\overline{L}$  together with a family of morphisms  $\overline{p}_\beta : \overline{L} \rightarrow L_\beta$  such that for any object  $L$  and any family of morphisms  $p_\beta : L \rightarrow L_\beta$  there exists a unique morphism  $\theta : L \rightarrow \overline{L}$  satisfying  $p_\beta = \overline{p}_\beta \circ \theta$ ; coproducts being products in the opposite category. Then :

- [4.3.1]  $i_\beta \dashv \Pi_\beta \dashv j_\beta$  and  $\sigma_\beta \dashv I_\beta \dashv \rho_\beta$ ;
- [4.3.2]  $\Pi_\beta$  is the product in BPos, JCLatt and MCLatt;
- [4.3.3] The coproducts in BPos, JCLatt, MCLatt are  $I_\beta$ ,  $i_\beta$ ,  $j_\beta$ .

In particular,  $\Pi_\beta$  preserves both the join and the meet,  $i_\beta$  preserves the join and  $j_\beta$  preserves the meet. Further,  $i_\beta$  preserves non-empty meets however  $i_\beta(1) \neq 1$ , whereas  $j_\beta$  preserves non-empty joins however  $j_\beta(0) \neq 0$ . We then have simple examples of weak morphisms. On the other hand,  $I_\beta$  preserves both the join and the meet,  $\sigma_\beta$  preserves the join and  $\rho_\beta$  preserves the meet. Further,  $\sigma_\beta(1) = 1$  however  $\sigma_\beta$  does not preserve binary meets, whereas  $\rho_\beta(0) = 0$  however  $\rho_\beta$  does not preserve binary joins. We then have simple examples of balanced morphisms.

Finally, in applications one is often led to consider maps  $f : L_1 \rightarrow L_2$  which preserve non-empty joins or maps  $g : L_2 \rightarrow L_1$  which preserve non-empty meets. In this way we obtain the categories WJCLatt and WMCLatt. While such maps do not possess strict Galois adjoints, by considering either codomain restrictions or pointed extensions we can nevertheless introduce the notion of weak adjunctions. Explicitly, let  $g : L_2 \rightarrow L_1$  preserve non-empty meets but not necessarily satisfy  $g(1_2) = 1_1$ . Then, for  $L^u = L \dot{\cup} \{1\}$  the upper pointed extension with adjoined universal maximal element  $a < 1$ , the two maps  $g^p : L_2 \rightarrow [0_1, g(1_2)] : a_2 \mapsto g(a_2)$  and  $g^u : L_2^u \rightarrow L_1^u : a_2 \mapsto g(a_2); \ 1 \mapsto 1$  do preserve arbitrary meets, and so have respective left adjoints  $f^p : [0_1, g(1_2)] \rightarrow L_2$  and  $f^u : L_1^u \rightarrow L_2^u$ . Now  $f^p$  is of the form  $\alpha_1 : [0_1, a_1] \rightarrow L_2$  whereas  $g^u$  is dense,  $g^u(a_2) = 1 \Leftrightarrow a_2 = 1$ , so that  $f^u$  is balanced,  $f^u(1) = 1$ . We are then led to consider the categories PJCLatt with sectional partial morphisms,  $PJ(L_1, L_2) = \cup_{a_1 \in L_1} J([0_1, a_1], L_2)$ , and UJCLatt with upper pointed morphisms,  $UJ(L_1, L_2) = BJ(L_1^u, L_2^u)$ . Note that for maps  $\alpha_1 : [0_1, a_1] \rightarrow L_2$  and  $\alpha_2 : [0_2, a_2] \rightarrow L_3$  we have  $(\alpha_2 \circ \alpha_1) : [0_1, \alpha_1^*(a_2)] \rightarrow L_3$ , since  $\alpha_1(x_1) \in [0_2, a_2] \Leftrightarrow \alpha_1(x_1) < a_2 \Leftrightarrow x_1 < \alpha_1^*(a_2)$ . Now, given morphisms  $\alpha \in PJ(L_1, L_2)$  and  $F \in UJ(L_1, L_2)$  let us define the maps

$$\begin{aligned} F_\alpha &: L_1^u \rightarrow L_2^u : x_1 \mapsto \alpha(a_1) \ (x_1 < a_1); \ 1 \ (x_1 \not< a_1); \\ G_\alpha &: L_2^u \rightarrow L_1^u : x_2 \mapsto \alpha^*(x_2); \ 1 \mapsto 1; \end{aligned}$$

$$\begin{aligned} \alpha_F &: [0_1, F^*(1_2)] \rightarrow L_2 : x_1 \mapsto F(x_1); \\ \beta_F &: L_2 \rightarrow [0_1, F^*(1_2)] : x_2 \mapsto F^*(x_2). \end{aligned}$$

For  $g \in WM(L_1, L_2)$  we then obtain :

[4.4.1]  $F_\alpha \dashv G_\alpha$  with  $F_{\alpha_2 \circ \alpha_1} = F_{\alpha_2} \circ F_{\alpha_1}$  and  $F_{\alpha_F} = F$ ;

[4.4.2]  $\alpha_F \dashv \beta_F$  with  $\alpha_{F_2 \circ F_1} = \alpha_{F_2} \circ \alpha_{F_1}$  and  $\alpha_{F_\alpha} = \alpha$ ;

[4.4.3] If  $\alpha \dashv g^p$  and  $F \dashv g^u$  then  $\alpha = \alpha_F$  and  $F = F_\alpha$ .

The first and second results enable us to pass from partial morphisms to pointed morphisms and conversely in a functorial manner, the third result implying that the two methods of defining weak adjoints are equivalent. In particular, the three categories  $\underline{\text{PJCLatt}}$ ,  $\underline{\text{WMCLatt}}^{\text{op}}$  and  $\underline{\text{UJCLatt}}$  are isomorphic.

## 5. Monads and closure operators

Recall that a monad on the category  $\underline{X}$  is a triple  $(T, \eta, \mu)$  consisting of an endofunctor  $T$  together with natural transformations  $\eta : \text{Id} \rightarrow T$  and  $\mu : T \circ T \rightarrow T$  satisfying the coherence conditions:  $\mu \circ T\mu = \mu \circ \mu T$ ;  $\mu \circ T\eta = \text{id}T$ ;  $\mu \circ \eta T = \text{id}T$ . Note that if  $L \dashv R$  is an adjunction via the natural transformations  $\eta$  and  $\varepsilon$  then  $(R \circ L, \eta, R\varepsilon L)$  is a monad. On the other hand, if  $(T, \eta, \mu)$  is a monad then setting  $\text{Ob}(\underline{X}^T)$  to be the set of  $T$ -algebras  $(A, \alpha)$ , where  $\alpha : TA \rightarrow A$  satisfies  $\alpha \circ \eta_A = \text{id}_A$  and  $\alpha \circ T\alpha = \alpha \circ \mu_A$ , and  $\text{Hom}((A, \alpha), (B, \beta))$  to be the set of  $T$ -morphisms  $f : A \rightarrow B$ , where  $f \circ \alpha = \beta \circ Tf$ , we obtain the so-called Eilenberg-Moore category generated by the adjunction  $F^T \dashv U^T$ , where  $F^T : A \mapsto (TA, \mu_A)$ ;  $f \mapsto Tf$  and  $U^T : (A, \alpha) \mapsto A$ ;  $f \mapsto f$ . Further, for any monad of the form  $T = R \circ L$  there exists a unique functor  $K : \underline{Y} \rightarrow \underline{X}^T$  such that  $R = U^T \circ K$  and  $F^T = K \circ L$ . Explicitly,  $K : A \mapsto (RA, R\varepsilon_A)$ ;  $f \mapsto Rf$ . Now in the context of posets considered as thin categories the coherence conditions are trivially satisfied, since any diagram which can be written must commute. An isotone map  $T : L \rightarrow L$  is then a monad if and only if the natural transformations  $\eta$  and  $\mu$  exist, which is the case if and only if  $\text{id} < T$  and  $T \circ T < T$  so that  $T$  is a closure operator. First, there exists a morphism  $\alpha : Ta \rightarrow a$  if and only if  $Ta < a$ , which is the case if and only if  $Ta = a$ . The category  $L^T$  is then the set of fixed points of  $T$ . Second, since  $F^T \dashv U^T$  we have that  $F^T$  preserves colimits and  $U^T$  preserves limits. In particular, if  $L$  is a complete lattice then so is  $L^T$ , with  $\bigwedge_T A = \bigwedge A$  and  $\bigvee_T A = T(\bigvee A)$ . Third, if  $f \dashv g$  then  $f \circ g \circ f = f$  and  $g \circ f \circ g = g$ , so that the Eilenberg-Moore category of the monad  $g \circ f$  is given by the image of  $g$ .

Recall that an atom of the lattice  $L$  is a minimal nonzero element and that the complete lattice  $L$  is called atomistic if  $a = \bigvee \{p \in \Sigma_L \mid p < a\}$  for each  $a \in L$ , where  $\Sigma_L$  is the set of atoms of  $L$ . A closure operator  $T : L \rightarrow L$  on the atomistic lattice is then called simple if  $T(0) = 0$  and  $T(p) = p$  for each  $p \in \Sigma_L$ . Since the atoms are all fixed points, the Eilenberg-Moore category associated to any simple closure operator is also atomistic. As we shall now indicate, we then obtain an equivalence between suitable categories of complete atomistic lattices and closure spaces, namely sets  $\Sigma$  equipped with a simple closure operator on  $P(\Sigma)$ . Explicitly, let  $f \dashv g$  be an adjunction between complete atomistic lattices and  $\alpha : \Sigma_1 \setminus K_1 \rightarrow \Sigma_2$  be a partially defined map between closure spaces. Then:

- [5.1.1]  $f(\Sigma_{L_1}) \subseteq \Sigma_{L_2} \cup \{0_2\}$  iff  $(\forall p_1 \in \Sigma_{L_1})(\exists p_2 \in \Sigma_{L_2}) p_1 < g(p_2)$ ;
- [5.1.2]  $\alpha(T_1 A_1 \setminus K_1) \subseteq T_2 \alpha(A_1 \setminus K_1)$  iff  $K_1 \cup \alpha^{-1}(A_2)$  is closed for  $A_2$  closed;
- [5.1.3] The above conditions are stable under composition.

We then obtain the dual categories JCALatt and MCALatt together with the category CSpace of closure spaces. Next, given morphisms  $\alpha \in CS((\Sigma_1, T_1), (\Sigma_2, T_2))$  and  $f \in JA(L_1, L_2)$  let us define

$$\begin{aligned} f_\alpha &: P(\Sigma_1)^{T_1} \rightarrow P(\Sigma_2)^{T_2} : A_1 \mapsto T_2 f(A_1 \setminus K_1), \\ g_\alpha &: P(\Sigma_2)^{T_2} \rightarrow P(\Sigma_2)^{T_1} : A_2 \mapsto K_1 \cup f^{-1}(A_2), \\ \alpha_f &: \Sigma_1 \setminus \{p_1 \in \Sigma_1 \mid f(p_1) = 0_2\} \rightarrow \Sigma_2 : p_1 \mapsto f(p_1). \end{aligned}$$

Further, let  $i : L \rightarrow P(\Sigma_L) : a \mapsto \{p \in \Sigma \mid p < a\}$  and  $\pi : P(\Sigma_L) \rightarrow L : A \mapsto \bigvee A$ , with  $\mathcal{L} : (\Sigma, T) \mapsto P(\Sigma)^T$ ;  $\alpha \mapsto f_\alpha$  and  $\mathcal{C} : L \mapsto (\Sigma_L, i \circ \pi)$ ;  $f \mapsto \alpha_f$ . We obtain:

- [5.2.1]  $\pi \dashv i$  with associated simple closure  $i \circ \pi : A \mapsto \{p \in \Sigma \mid p < \bigvee A\}$ ;
- [5.2.2]  $\alpha_f$  is a morphism of closure spaces;
- [5.2.3]  $f_\alpha \dashv g_\alpha$  is an adjunction of complete atomistic lattices;
- [5.2.4]  $\mathcal{L} \dashv \mathcal{C}$  is an equivalence between CSpace and JCALatt.

Note that these results restrict in a canonical manner to complete atomistic ortholattices. Indeed, for each such lattice let us define the binary relation  $\perp$  on  $\Sigma_L$  by  $p \perp q$  if  $p < q'$ . Then the induced map  $T : P(\Sigma_L) \rightarrow P(\Sigma_L) : A \mapsto A^{\perp\perp}$  with  $A^\perp = \{q \in \Sigma \mid (\forall p \in A) p \perp q\}$  is a simple closure operator, with  $A \subseteq \Sigma_L$  biorthogonal if and only if  $A = \{p \in \Sigma_L \mid p < \bigvee A\}$ . Now  $\perp$  is symmetric,  $p \perp q \Rightarrow q \perp p$ , antireflexive,  $p \perp q \Rightarrow p \neq q$ , and separating,  $p \neq q \Rightarrow (\exists r \in \Sigma_L) p \perp r \text{ \& } q \not\perp r$ . In fact the last condition is exactly the requirement that singletons be biorthogonal. We then obtain an equivalence between the category JCAoLatt of atomistic join complete ortholattices and the category OSpace of orthogonality spaces.

The above considerations allow us to recover some standard results concerning power functors and complete Boolean algebras, results which will be generalised in the next section to transition structures. First, let CaBAlg be the category of complete atomic Boolean algebras with as morphisms all maps preserving the join, the meet, and the orthocomplementation. Note that any two of these conditions implies the third. Indeed,  $\bigwedge A = (\bigvee A')'$  and  $\bigvee A = (\bigwedge A')'$  so that any map which preserves the join and the orthocomplementation also preserves the meet, whereas any map which preserves the meet and the orthocomplementation also preserves the join. On the other hand, if  $f$  preserves finite joins and meets and satisfies  $f(0_1) = 0_2$  and  $f(1_1) = 1_2$ , then  $f(a_1) \wedge f(a'_1) = f(a_1 \wedge a'_1) = f(0_1) = 0_2$  and  $f(a_1) \vee f(a'_1) = f(a_1 \vee a'_1) = f(1_1) = 1_2$ , so that  $f(a'_1) = f(a_1)'$  since any Boolean algebra is uniquely complemented. Second, given the set function  $\alpha : \Sigma_1 \rightarrow \Sigma_2$  define  $S_\alpha : P(\Sigma_2) \rightarrow P(\Sigma_1) : A_2 \mapsto \{x_1 \in \Sigma_1 \mid \alpha(x_1) \in A_2\}$  and  $P_\alpha : P(\Sigma_1) \rightarrow P(\Sigma_2) : A_1 \mapsto \{x_2 \in \Sigma_2 \mid (\exists x_1 \in A_1) \alpha(x_1) = x_2\}$ . Third, for  $B$  a complete atomistic Boolean algebra let  $\mu_B : B \rightarrow P(\Sigma_B) : a \mapsto \{p \in \Sigma_B \mid p < a\}$

and  $\rho_B : P(\Sigma_B) \rightarrow B : A \mapsto \bigvee A$ . Then for  $f \dashv g$  an adjunction between complete atomistic Boolean algebras we obtain :

[5.3.1]  $P_\alpha \dashv S_\alpha$ , with  $\alpha$  injective iff  $S_\alpha \circ P_\alpha = \text{id}$  or surjective iff  $P_\alpha \circ S_\alpha = \text{id}$ ;

[5.3.2]  $\mu_B \dashv \rho_B$  is an equivalence, and  $f(\Sigma_{B_1}) \subseteq \Sigma_{B_2}$  iff  $g(a'_2) = g(a_2)'$ .

The first result implies that the maps  $S : \underline{\text{Set}} \rightarrow \underline{\text{CaBalg}}^{\text{op}} : \Sigma \mapsto P(\Sigma)$ ;  $\alpha \mapsto S_\alpha$  and  $P : \underline{\text{Set}} \rightarrow \underline{\text{JCLatt}} : \Sigma \mapsto P(\Sigma)$ ;  $\alpha \mapsto P_\alpha$  are functors. Indeed,  $S$  is well defined,  $x_1 \in S_\alpha(A_2^c) \Leftrightarrow \alpha(x_1) \in A_2^c \Leftrightarrow \alpha(x_1) \notin A_2 \Leftrightarrow x_1 \notin S_\alpha(A_2) \Leftrightarrow x_1 \in S_\alpha(A_2)^c$ , and functorial, since we have  $x \in S_{\text{id}}(A) \Leftrightarrow (\exists y \in A) x = \text{id}(x) = y \Leftrightarrow x \in A$  and  $x_1 \in S_{\alpha_2 \circ \alpha_1}(A_3) \Leftrightarrow (\alpha_2 \circ \alpha_1)(x_1) \in A_3 \Leftrightarrow \alpha_1(x_1) \in S_{\alpha_2}(A_3) \Leftrightarrow x_1 \in (S_{\alpha_1} \circ S_{\alpha_2})(A_3)$ . The second result implies that  $S$  is an equivalence: by the first part the equivalence between closure spaces and complete atomistic lattices with join preserving atomic maps restricts to an equivalence between sets and complete atomistic Boolean algebras with join preserving kernel free atomic maps, whereas by the second part this latter category is dual to the category of complete atomistic Boolean algebras with maps preserving the meet, the orthocomplement, and so the join.

## 6. Transition structures

Now, in applying the power construction to complete lattices we have two relevant orders, namely  $A \subseteq B$  in  $P(L)$  and  $a < b$  in  $L$ . It is then of interest to consider the strong preorder on  $P(L)$  defined by  $A \ll B$  if  $\bigvee A < \bigvee B$ . Note that this preorder is indeed superordinate to both of the original orders, since we have that  $A \subseteq B \Rightarrow \bigvee A < \bigvee B \Rightarrow A \ll B$  and  $a < b \Rightarrow \bigvee \{a\} = a < b = \bigvee \{b\} \Rightarrow \{a\} \ll \{b\}$ . Further, in the context of join preserving maps the minimal element may be treated as redundant, since all such maps satisfy the condition  $f(0_1) = 0_2$ . We are then lead to consider the truncated power set  $P_0(L) = P(L \setminus \{0\})$ . This reduction is reasonable, since  $P(L) = P((L \setminus \{0\}) \cup \{0\}) = P(L \setminus \{0\}) \times P(\{0\})$  and  $\bigvee(A \cup \{0\}) = (\bigvee A) \vee (\bigvee \{0\}) = (\bigvee A) \vee 0 = \bigvee A$ . In particular, we obtain the category PStruct of power structures,  $PS(L_1, L_2) = P_0(J(L_1, L_2))$ , and the category FStruct of functional structures,  $FS(L_1, L_2) = J(P_0(L_1), P_0(L_2))$ . Now in the category of power structures we focus on the set of join preserving maps  $f : L_1 \rightarrow L_2$ , whereas in the category of functional structures we focus on the set of union preserving maps  $\theta : P_0(L_1) \rightarrow P_0(L_2)$ . Since the condition of join preservation may be written  $f(\bigvee A_1) = \bigvee P_f(A_1)$ , it is then of interest to consider intermediate structures defined by pairs  $(f, \theta)$  satisfying the coherence condition  $f \succ \theta \Leftrightarrow f(\bigvee A_1) = \bigvee \theta(A_1)$ , i.e., for the operational resolution  $J : P_0(L) \rightarrow L : A \mapsto \bigvee A$  we have  $f \succ \theta \Leftrightarrow f \circ J_1 = J_2 \circ \theta$ . Then, for  $\ell : L \rightarrow P_0(L) : a \mapsto (0, a]$ :

[6.1.1]  $J \dashv \ell$  with  $J \circ \ell = \text{id}$ ;

[6.1.2] Given union preserving  $\theta$ , if  $f \succ \theta$  then  $f$  is unique and join preserving;

[6.1.3] If  $\theta$  preserves unions, there exists  $f \succ \theta$  iff  $\theta$  is strongly isotone;

[6.1.4] If  $f_1 \succ \theta_1$ ,  $f_2 \succ \theta_2$  then  $f_2 \circ f_1 \succ \theta_2 \circ \theta_1$ ; if  $f_\alpha \succ \theta_\alpha$  then  $\bigvee_\alpha f_\alpha \succ \bigcup_\alpha \theta_\alpha$ .

The first result is a trivial technical lemma allowing the compact presentation of the coherence condition. The second and third results allow the reconstruction of the underlying morphism  $f$  associated to a given covering  $\theta$ . The first part of the fourth result allows us to introduce the category  $\mathbf{TStruct}$  of transition structures, where  $TS(L_2, L_2)$  is the set of pairs  $(f, \theta)$  such that  $f \succ \theta$ , whereas the second part allows us to define the subcategory  $\mathbf{BStruct}$  of based structures, whose morphisms are obtained by closing the Hom-sets of  $\mathbf{PStruct}$  with respect to unions. We then obtain an inclusion hierarchy of intermediate categories between  $\mathbf{PStruct}$ , construed as an isomorphic image of  $\mathbf{JCLatt}$ , and  $\mathbf{FStruct}$ , construed as the maximal power construction on lattices.

Note that the obvious inclusions  $\mathbf{PStruct} \hookrightarrow \mathbf{BStruct} \hookrightarrow \mathbf{TStruct} \hookrightarrow \mathbf{FStruct}$  are functorial, since the objects and compositions laws are the same in the four categories. Further, the quantaloid morphism  $F : \mathbf{TStruct} \rightarrow \mathbf{JCLatt} : (f, \theta) \mapsto f$  is retractive, since  $F \circ I = \text{id}$  for  $I : \mathbf{JCLatt} \rightarrow \mathbf{TStruct} : f \mapsto (f, P_f)$ . Finally, the above quantaloid inclusions are all strict, the category of transition structures then being the maximal faithful coherent enrichment of  $\mathbf{JCLatt}$ . Explicitly, let  $\mathbf{2}$  be the two element lattice so that  $P_0(\mathbf{2}) = P(\mathbf{2} \setminus \{0\}) = P(\{1\})$ . We then obtain :

$$[6.2.1] \quad PS(\mathbf{2}, L) = L \text{ whereas } BS(\mathbf{2}, L) = TS(\mathbf{2}, L) = FS(\mathbf{2}, L) = P_0(L);$$

$$[6.2.2] \quad PS(L, \mathbf{2}) = BS(L, \mathbf{2}) = TS(L, \mathbf{2}) = L \text{ whereas } FS(L, \mathbf{2}) = P_0(L);$$

$$[6.2.3] \quad \text{The inclusions } \mathbf{PStruct} \hookrightarrow \mathbf{BStruct} \hookrightarrow \mathbf{TStruct} \hookrightarrow \mathbf{FStruct} \text{ are strict.}$$

These results can be directly extended to partial constructs, namely categories  $\mathcal{A}$  which are concrete over the category  $\mathbf{PSet}$  of sets with partially defined maps. Indeed, let us consider a partial construct  $U : \mathcal{A} \rightarrow \mathbf{PSet}$  such that each Hom-set contains a non-trivial morphism,  $\text{Ker}(Uf) \neq \text{Dom}(Uf)$ . First, the image of  $\mathcal{P} \circ U$ , where  $\mathcal{P} : \mathbf{PSet} \rightarrow \mathbf{JCLatt}$  is the partial power functor, defines a category  $\mathcal{PA}$  which generalises the category  $\mathbf{PStruct}$  of power structures. Second, by analogy to the category  $\mathbf{BStruct}$  of based structures, closing  $\mathcal{PA}$  Hom-sets under pointwise unions we obtain the quantaloid  $\mathcal{Q}^-\mathcal{A}$ . Third, by analogy to the category  $\mathbf{FStruct}$  of functional structures, taking all union preserving maps we obtain the quantaloid  $\mathcal{Q}^+\mathcal{A}$ . Note that functors  $F : \mathcal{A} \rightarrow \mathcal{B}$  canonically lift to  $\mathcal{P}$  and  $\mathcal{Q}^-$  but not to  $\mathcal{Q}^+$ . In a certain sense, then, we may consider  $\mathcal{Q}^-\mathcal{A}$  as a functorial enrichment of  $\mathcal{A}$  and  $\mathcal{Q}^+\mathcal{A}$  as its contextual enrichment. Now, the above inclusion hierarchy arises for the case  $U : \mathbf{JCLatt} \rightarrow \mathbf{PSet} : L \mapsto L \setminus \{0\}$ . In particular, for any subquantaloid  $\mathcal{A}$  of  $\mathbf{JCLatt}$  we may generalise the category  $\mathbf{TStruct}$  of transition structures to the quantaloid  $\mathcal{Q}^0\mathcal{A}$ , the map  $f$  being the property transition associated to the state transition  $\theta$ . In fact, given any subcategory  $\mathcal{A}$  of  $\mathbf{JCLatt}$ , considering  $\mathcal{PA}$  as a subcategory of  $\mathbf{TStruct}$  we can define  $\mathcal{EA}$ , the smallest subquantale of  $\mathbf{JCLatt}$  containing  $\mathcal{A}$ , as the image of  $\mathcal{PA}$  under the underlying functor. We then obtain pre-enrichments as free extensions of subcategories guaranteeing that all  $\mathcal{Q}^-$ -morphisms may be considered as state transitions.

## 7. States and properties

In this section we briefly indicate how the above categorical techniques find a direct application in the so-called ‘Geneva School approach’, a framework theory allowing abstract mathematical representations of concrete physical notions. The primitive concrete notions of this approach are those of ‘particular physical system’, namely a part of the ostensibly external phenomenal world considered as distinct from its surroundings, and ‘definite experimental project’, namely a real experimental procedure where we have decided in advance what would be the positive result should we perform the experiment. We then obtain the mathematical notions of ‘state’, construed as an abstract name for a singular realisation of the physical system, and ‘property’, construed as the element of reality corresponding to a definite experimental project. Now the sets  $\Sigma$  of states and  $L$  of properties each possess mathematical structure arising directly from their physical natures. First, two states  $\mathcal{E}$  and  $\mathcal{E}'$  are called orthogonal, written  $\mathcal{E} \perp \mathcal{E}'$ , if there exists a definite experimental project which is certain for the first and impossible for the second. The orthogonality relation is then trivially symmetric and antireflexive. In particular, writing  $A^\perp$  for the orthogonal of  $A \subseteq \Sigma$  we have that  $A \subseteq A^{\perp\perp}$  and  $A \subseteq B \Rightarrow B^\perp \subseteq A^\perp$ . The map  $A \mapsto A^{\perp\perp}$  is then a closure operator, so that the set  $(\Sigma, \perp)$  of biorthogonals is a complete atomistic ortholattice with atoms  $\{\mathcal{E}\}^{\perp\perp}$ . Second, the property  $a$  is called stronger than the property  $b$ , written  $a < b$ , if  $b$  is actual whenever  $a$  is actual. The strength relation is then trivially a partial order. Further, the construction of the product  $\Pi \mathbf{A}$  of the family  $\mathbf{A}$  of definite experimental projects implies that  $L$  is a complete lattice whose meet is given by semantic conjunction. These two structures are intimately linked by the so-called Cartan maps, which associate to each property  $a$  the set  $\mu(a)$  of states in which it is actual and to each state  $\mathcal{E}$  the set  $S(\mathcal{E})$  of its actual properties. Indeed, by definition  $\mu : L \rightarrow P(\Sigma)$  is injective and satisfies  $\mu(\bigwedge A) = \bigcap \mu(A)$ , whereas the induced map  $\rho : \Sigma \rightarrow L : \mathcal{E} \mapsto \bigwedge S(\mathcal{E})$  satisfies  $a < b \Leftrightarrow [\rho(\mathcal{E}) < a \Rightarrow \rho(\mathcal{E}) < b]$ .

The standard axioms then allow one to characterise the images of  $\mu$  and  $\rho$ , namely those sets of states which represent properties and those properties which represent states. First, note that each atom  $p$  is a state representative. Indeed,  $p \neq 0$  so that there exists a state  $\mathcal{E}$  in which  $p$  is actual. Then  $\rho(\mathcal{E}) < p$  so that  $p = \rho(\mathcal{E})$ . Further, operationally the state represents all that can be done with certainty with the system. The general principle that the generation of any actual property requires the destruction of another then indicates that we should postulate a bijective correspondence between state representatives and atoms. The property lattice is then atomistic. Second, the orthogonality of two states is an objective feature of the system as a whole. We are then led to suppose the existence of a map  $^\# : \Sigma \rightarrow L : p \mapsto p^\#$  such that  $p \perp q$  if and only if  $q < p^\#$ , with induced action  $' : L \rightarrow L : a \mapsto \bigwedge \{p^\# \mid p < a\}$ . Then  $\mu(a') = \mu(a)^\perp$ , and  $'$  satisfies the conditions  $a < b \Rightarrow b' < a'$  and  $a < a''$ . Third, we can identify the lattices  $(\Sigma, \perp)$  and  $L$  by supposing in addition that  $'$  be surjective, that is, that each property have an opposite. Then the map  $'$  is an orthocomplementation, and if  $p \neq q$  there exists  $r$  such that  $p \perp r$  and  $q \not\perp r$ . In summary, with these three axioms we

can model the set of properties of any given physical system by a complete atomistic ortholattice, and the set of its states by an orthogonality space, the physical state-property duality being a concrete realisation of the abstract equivalence between the categories OSpace and JCAoLatt. Note that any complete atomistic ortholattice can be putatively interpreted as a property lattice. First, in order to interpret atoms as states, we assume that for each  $p \in \Sigma$  there exists a definite experimental project  $\alpha_p$  which is certain for  $q$  if and only if  $q = p$ . Second, in order to interpret  $q < p'$  as an orthogonality relation, we assume that for each  $p \in \Sigma$  there exists a definite experimental project  $\beta_p$  which is certain for  $q$  if and only if  $q < p'$ . To obtain a coherent interpretation we then assume that  $\beta_p = \alpha_{\tilde{p}}$ , the inverse obtained by exchanging the terms of the alternative. Completing with respect to the product we then obtain exactly the lattice  $L$ , since the condition  $a = \bigwedge \{ p' \mid p < a \}$  implies that  $a'$  is generated by the  $\beta_p$  with  $p$  majorised by  $a$ . For a somewhat different categorical implementation of the Geneva School axioms see [Valckenborgh 2000].

Apart from its mathematical elegance, the categorical realisation of state-property duality is a powerful tool in the study of axiomatic quantum theory. For example, it is implicit in the construction of many standard secondary notions, such as the classical decomposition, construed as a product in JCAoLatt, and observables, construed as morphisms in COLatt with domain a complete Boolean algebra whose Stone space encodes the measurement scale. First,  $z \in L$  is called central if there exists a direct product decomposition  $\pi : L_1 \times L_2 \simeq L$  with  $z = \pi(1, 0)$ , in which case  $z$  has unique complement  $z' = \pi(0, 1)$ . Now for complete atomistic ortholattices, the central elements are exactly the classical properties, namely those  $z \in L$  such that for each  $p \in \Sigma_L$  either  $p < z$  or  $p < z'$ . In particular, for  $\alpha \in \Omega$  the atoms of the center of  $L$  we have that  $L = \times_{\alpha} [0, \alpha]$ . Second, the spectrum of the observable  $\mu : \mathbf{B} \rightarrow L$  may be obtained from the decomposition  $N = \mu^*(0)$ ,  $D = \bigvee \{ E \in \Sigma_{\mathbf{B}} \mid E < N' \}$ ,  $C = D' \wedge N'$ , the interval  $[0, D]$  being atomistic and so representing the discrete spectrum and the interval  $[0, C]$  being atomless and so representing the continuous spectrum. Further, different physical and mathematical aspects of a given problem are often best treated using different techniques, so that the various categorical equivalences exposed above are crucial in enabling natural translations. For example, the meet has a definite physical interpretation as semantic conjunction, being operationally constructible via the product, whereas it is often technically useful to consider join preserving maps, these being the usual objects for representation theorems. The duality between MCLatt and JCLatt then allows one to define a concept in physically meaningful terms and then study it using mathematically adequate techniques. For example, the construction of Hilbertian realisations for Arguesian orthogeometries  $(G, \oplus)$  is purely categorical, being based on an appropriate embedding of  $G$  as a hyperplane in the projective geometry  $\overline{G}$  whose elements are endomorphisms of  $G$  with a given fixed axis  $H$ . First, the set  $V = \overline{G} \setminus G$  has a natural vector space structure over the division ring of homotheties defined with respect to a fixed  $0 \in V$ . Second, each non-degenerate morphism  $g : G_1 \setminus N_1 \rightarrow G_2$  admits a canonical extension  $h : \overline{G}_1 \setminus \overline{N}_1 \rightarrow \overline{G}_2$  whose restriction to  $V_1$  defines a semilinear

map  $f : V_1 \rightarrow V_2$ . Third, and finally, the orthogonality relation defines a non-degenerate homomorphism  $g : G \rightarrow G^* : p \mapsto \{p\}^\perp$ , where  $G^*$  is the dual geometry of hyperplanes of  $G$ , whose associated quasilinear map  $f : V \rightarrow V^*$  then defines a Hermitian form.

A case in point is the notion of deterministic evolutions. First, one can partially encode an imposed evolution by defining a map  $\Phi_{01}$  which associates to each definite experimental project  $\alpha_1$  defined at the final time  $t_1$  the definite experimental project  $\alpha_0$  defined at the initial time  $t_0$  by the prescription ‘Evolve the system as required and effectuate  $\alpha_1$ ’ [Daniel 1982, 1989]. Then it can be physically demonstrated that  $\Phi_{01}$  preserves the product and maps  $O_1$  to  $O_0$ . We thereby obtain an induced map  $\varphi_{01} : L_1 \rightarrow L_0$  which preserves non-empty meets and maps  $0_1$  to  $0_0$ . In words, each given evolution induces a weak balanced meet morphism. In the simplest case, where  $\varphi_{01}(1_1) = 1_0$  so that the system is never destroyed by the evolution, we then obtain the dense right adjoint  $\psi_{10} : L_0 \rightarrow L_1$ . Physically, the adjunction condition implies that  $\psi_{10}(p_0)$  is the strongest final property whose actuality is guaranteed by the evolution for the initial state defined by the atom  $p_0$ . If this property in fact determines the final state of the system, then we can realise the evolution as a dense atomic join morphism. Finally, under suitable stability conditions two orthogonal final states arise from two orthogonal initial states, since if  $\alpha_1$  separates  $\psi_{10}(p_0)$  and  $\psi_{10}(q_0)$  then  $\Phi_{01}\alpha_1$  separates  $p_0$  and  $q_0$ . In the Hilbertian context, one can then prove that a sufficiently continuous evolution may be represented by a unitary flow [Faure, Moore and Piron 1995]. More generally, the cognitive duality between causal assignment and consecutive propagation of properties may be encoded in the quantaloid isomorphism  $A^* : \underline{\mathbf{JCLatt}} \rightarrow \underline{\mathbf{MCLatt}}^{\text{coop}}$  [Coecke 2000; Coecke, Moore and Stubbe 2000]. Explicitly, given two property lattices  $L_1$  and  $L_2$  let us write  $a_1 \rightarrow a_2$  if  $a_1 \in L_1$  is a material cause of  $a_2 \in L_2$ . Then by the operational signification of the lattice partial order as semantic implication and the operational construction of the lattice meet as semantic conjunction, the relation  $\rightarrow$  should be fully isotone,  $(x_1 < a_1 \rightarrow a_2 < x_2) \Rightarrow (x_1 \rightarrow x_2)$ , and preserve right non-empty meets,  $(a_1 \rightarrow a_{2\alpha}) \Rightarrow (a_1 \rightarrow \bigwedge_\alpha a_{2\alpha})$ . Setting  $g : L_2 \rightarrow L_1 : a_2 \mapsto \bigvee \{ a_1 \in L_1 \mid a_1 \rightarrow a_2 \}$  we may then express causal assignment by a weak meet morphism, in the sense that  $(a_1 \rightarrow a_2) \Leftrightarrow (a_1 < g(a_2))$ , whose pseudoadjoint  $f : L_1^u \rightarrow L_2^u$  implements the notion of consecutive propagation of properties. In particular, the interpretation of propagation as an evolutive flow may be complemented by the interpretation of assignment in terms of states of compoundness.



## APPENDIX

In this appendix we provide proofs of the results cited in the text.

**Proof of proposition 3.1 :**

- (1) Let  $\text{id}_1 < (g \circ f)$  and  $(f \circ g) < \text{id}_2$ . Then  $f(a_1) < a_2 \Rightarrow a_1 < (g \circ f)(a_1) < g(a_2)$  and  $a_1 < g(a_2) \Rightarrow f(a_1) < (f \circ g)(a_2) < a_2$ . Let  $f(a_1) < a_2 \Leftrightarrow a_1 < g(a_2)$ . Then for each  $a_1 \in L_1$  we have that  $f(a_1) < f(a_1) \Rightarrow a_1 < (g \circ f)(a_1)$  and for each  $a_2 \in L_2$  we have that  $g(a_2) < g(a_2) \Rightarrow (f \circ g)(a_2) < a_2$ .
- (2) First  $f^{-1} = g \Leftrightarrow g \circ f = \text{id}_1$ ; second  $f \circ g = \text{id}_2 \Leftrightarrow \text{id}_1 < g \circ f, f \circ g < \text{id}_2$ ; third  $\text{id}_2 < f \circ g, g \circ f < \text{id}_1 \Leftrightarrow f \dashv g \dashv f$ .
- (3)  $\text{id}_1 < g \circ f$  so that  $f = f \circ \text{id}_1 < f \circ g \circ f$  and  $g = \text{id}_2 \circ g < g \circ f \circ g$ , whereas  $f \circ g < \text{id}_2$  so that  $f \circ g \circ f < \text{id}_2 \circ f = f$  and  $g \circ f \circ g < g \circ \text{id}_2 = g$ .
- (4) Let  $f < \bar{f}$ . Then  $(f \circ \bar{g})(a_2) < (\bar{f} \circ \bar{g})(a_2) < a_2 \Rightarrow \bar{g}(a_2) < g(a_2)$  for each  $a_2 \in L_2$ . Let  $\bar{g} < g$ . Then  $a_1 < (\bar{g} \circ \bar{f})(a_1) < (g \circ \bar{f})(a_1) \Rightarrow f(a_1) < \bar{f}(a_1)$  for each  $a_1 \in L_1$ .
- (5)  $\text{id}(a) < b \Leftrightarrow a < b \Leftrightarrow a < \text{id}(b)$ , and so  $\text{id} \dashv \text{id}$ . Let  $f \dashv g$  and  $\bar{f} \dashv \bar{g}$ . Then  $(\bar{f} \circ f)(a_1) < a_3 \Leftrightarrow f(a_1) < \bar{g}(a_3) \Leftrightarrow a_1 < (g \circ \bar{g})(a_3)$ , and so  $(\bar{f} \circ f) \dashv (g \circ \bar{g})$ .

**Proof of proposition 3.2 :**

- (1)  $f(\bigvee A_1) < a_2 \Leftrightarrow \bigvee A_1 < g(a_2) \Leftrightarrow (\forall a_1 \in A_1) [a_1 < g(a_2) \Leftrightarrow f(a_1) < a_2] \Leftrightarrow \bigvee f(A_1) < a_2$ ;  $a_1 < g(\bigwedge A_2) \Leftrightarrow f(a_1) < \bigwedge A_2 \Leftrightarrow (\forall a_2 \in A_2) [f(a_1) < a_2 \Leftrightarrow a_1 < g(a_2)] \Leftrightarrow a_1 < \bigwedge g(A_2)$ .
- (2) If  $f(\bigvee A) = \bigvee f(A)$  then  $f(a_1) < a_2 \Rightarrow a_1 < \bigvee \{x_1 \in L_1 \mid f(x_1) < a_2\} = f^*(a_2)$  and  $a_1 < f^*(a_2) \Rightarrow f(a_1) < f(\bigvee \{x_1 \in L_1 \mid f(x_1) < a_2\}) = \bigvee \{f(x_1) \mid f(x_1) < a_2\} < a_2$ .
- (3) If  $g(\bigwedge A) = \bigwedge g(A)$  then  $a_1 < g(a_2) \Rightarrow g_*(a_1) = \bigwedge \{x_2 \in L_2 \mid a_1 < g(x_2)\} < a_2$  and  $g_*(a_1) < a_2 \Rightarrow a_1 < \bigwedge \{g(x_2) \mid a_1 < g(x_2)\} = g(\bigwedge \{x_2 \in L_2 \mid a_1 < g(x_2)\}) < g(a_2)$ .
- (4)  $(\bigvee_\alpha f_\alpha)(a_1) < a_2 \Leftrightarrow (\forall \alpha) f_\alpha(a_1) < a_2 \Leftrightarrow (\forall \alpha) a_1 < g_\alpha(a_2) \Leftrightarrow a_1 < (\bigwedge_\alpha g_\alpha)(a_2)$ .
- (5)  $(f \circ \bigvee_\alpha f_\alpha)(a_1) < a \Leftrightarrow (\forall \alpha) [f_\alpha(a_1) < g(a) \Leftrightarrow a_1 < (g_\alpha \circ g)(a)] \Leftrightarrow a_1 < \bigwedge_\alpha (g_\alpha \circ g)(a)$ ;  $(\bigvee_\alpha f_\alpha \circ f)(a) < a_2 \Leftrightarrow (\forall \alpha) [f(a) < g_\alpha(a_2) \Leftrightarrow a < (g \circ g_\alpha)(a_2)] \Leftrightarrow a < \bigwedge_\alpha (g \circ g_\alpha)(a_2)$ .

**Proof of proposition 3.3 :**

- (1) First, let  $A^* : J(L_1, L_2) \rightarrow M(L_2, L_1)^{\text{co}} : f \mapsto f^*$ . Then  $A^*$  is well defined, since  $f^*$  is unique and preserves meets, and isotone,  $f < \bar{f} \Rightarrow \bar{f}^* < f^* \Rightarrow f^* <^{\text{op}} \bar{f}^*$ . Second, let  $A_* : M(L_2, L_1)^{\text{co}} \rightarrow J(L_1, L_2) : g \mapsto g_*$ . Then  $A_*$  is well defined, since  $g_*$  is unique and preserves joins, and isotone,  $g <^{\text{op}} \bar{g} \Rightarrow \bar{g} < g \Rightarrow g_* < \bar{g}_*$ . Third,  $A_* \circ A^* = \text{id}$  and  $A^* \circ A_* = \text{id}$ . Indeed, if  $f \dashv g$  then  $g = f^*$  and  $f = g_*$ . Hence  $(A_* \circ A^*)(f) = (f^*)_* = g_* = f$  and  $(A^* \circ A_*)(g) = (g_*)^* = f^* = g$ .
- (2) First, let  $A^* : \underline{\text{JCLatt}} \rightarrow \underline{\text{MCLatt}}^{\text{op}} : L \mapsto L; f \mapsto f^*$ . Then  $A^*$  is well defined, since we have  $f \in J(L_1, L_2) \Rightarrow f^* \in M(L_2, L_1) = M^{\text{op}}(L_1, L_2)$ , and a functor,  $A^*(f_2 \circ f_1) = (f_2 \circ f_1)^* = f_1^* \circ f_2^* = f_2^* \circ^{\text{op}} f_1^* = A^*(f_2) \circ^{\text{op}} A^*(f_1)$ . Second, let  $A_* : \underline{\text{MCLatt}}^{\text{op}} \rightarrow \underline{\text{JCLatt}} : L \mapsto L; g \mapsto g_*$ . Then  $A_*$  is well defined, since we have  $g \in M^{\text{op}}(L_1, L_2) = M(L_2, L_1) \Rightarrow g_* \in J(L_2, L_2)$ , and a functor,

$A_*(g_2 \circ^{\text{op}} g_1) = A_*(g_1 \circ g_2) = (g_1 \circ g_2)_* = g_{2*} \circ g_{1*} = A_*(g_2) \circ A_*(g_1)$ . Third, as above  $A_* \circ A^* = \text{id}$  and  $A^* \circ A_* = \text{id}$ .

(3) Let  $A^* : \mathbf{JCLatt} \rightarrow \mathbf{MCLatt}^{\text{coop}} : f \mapsto f^*$  and  $A_* : \mathbf{MCLatt}^{\text{coop}} \rightarrow \mathbf{JCLatt} : g \mapsto g_*$ . Then combining the above two results we have that  $A^*$  and  $A_*$  are well defined, are quantaloid morphisms, and satisfy  $A_* \circ A^* = \text{id}$  and  $A^* \circ A_* = \text{id}$ .

### Proof of proposition 3.4:

(1)  $a_1 < b_1 \Rightarrow b'_1 < a'_1 \Rightarrow \alpha(b'_1) < \alpha(a'_1) \Rightarrow (C\alpha)(a_1) = \alpha(a'_1)' < \alpha(b'_1)' = (C\alpha)(b_1)$ , so that  $C(\alpha)$  is isotone and  $C$  is well defined.  $(C \text{id})(a) = \text{id}(a)' = a'' = a$  and  $C(\alpha_2 \circ \alpha_1)(a_1) = (\alpha_2 \circ \alpha_1)(a'_1)' = \alpha_2(\alpha_1(a'_1)'')' \alpha_2((C\alpha_1)(a_1)')' = (C\alpha_2 \circ C\alpha_1)(a_1)$ , so that  $C$  is a functor.  $(CC\alpha)(a_1) = (C\alpha)(a'_1)' = \alpha(a''_1)'' = \alpha(a_1)$ . If  $f \dashv g$  then  $(Cg)(a_2) = g(a'_2)' < a_1 \Leftrightarrow a'_1 < g(a'_2) \Leftrightarrow f(a'_1) < a'_2 \Leftrightarrow a_2 < f(a'_1)' = (Cf)(a_1)$ , so that  $C(g) \dashv C(f)$ . Hence  $C$  maps join maps to meet maps and conversely.

(2) First, if  $f \dashv g$  then  $f^\dagger = C(g) \dashv C(f) = g_\dagger$  and so  $\dagger$  is well defined on  $\mathbf{JCoLatt}$ . Second,  $(f_2 \circ f_1)^\dagger = C((f_2 \circ f_1)^*) = C(f_2^* \circ f_1^*) = C(f_1^*) \circ C(f_2^*) = f_1^\dagger \circ f_2^\dagger$  and  $f^{\dagger\dagger} = C(f^\dagger^*) = C(C(f)_*) = (CC)(f) = f$ , so that  $\dagger$  is an involution. Third, by the adjunction condition  $f^\dagger(a_2) < a'_1 \Leftrightarrow a_1 < f^\dagger(a_2)' = f^*(a'_2) \Leftrightarrow f(a_1) < a'_2$ . Fourth,  $f^\dagger \circ f = 0_1 \Leftrightarrow (f^\dagger \circ f)(1_1) < 0_1 \Leftrightarrow f(1_1) = f(0'_1) < f(1_1)' \Leftrightarrow f(1_1) = 0_2 \Leftrightarrow f = 0_2$ .

(3) If  $u^\dagger \circ u = \text{id}$  then  $a_1 < b'_1 \Leftrightarrow (u^\dagger \circ u)(a_1) = a_1 < b'_1 \Leftrightarrow u(b_1) < u(a_1)' \Leftrightarrow u(a_1) < u(b_1)'$ . If conversely then  $a_1 < x_1 \Leftrightarrow u(a_1) < u(x'_1)' \Leftrightarrow u(x'_1) < u(a_1)' \Leftrightarrow (u^\dagger \circ u)(a_1) < x_1$ .

(4) First,  $h_*(a'_2) < a_1 \Leftrightarrow a'_2 < h(a_1) \Leftrightarrow h(a'_1) = h(a_1)' < a_2 \Leftrightarrow a'_1 < h^*(a_2) \Leftrightarrow h^*(a'_2)' < a_1$ . Second,  $h^\dagger(a_2) = h^*(a'_2)' = h_*(a_2)'' = h_*(a_2)$  and  $h_\dagger(a_2) = h_*(a'_2)' = h^*(a_2)'' = h^*(a_2)$ . Third,  $h \circ h^\dagger \circ h = h \circ h_* \circ h = h$  and  $h \circ h_\dagger \circ h = h \circ h^* \circ h = h$ .

### Proof of proposition 4.1:

(1) First, if  $\chi = 0$  then  $0 < C^a(x)$  and  $\alpha_a(0) = 0 < x$  whereas if  $\chi = 1$  then  $1 < C^a(x) \Leftrightarrow C^a(x) = 1 \Leftrightarrow \alpha_a(1) = a < x$ . Hence  $\alpha_a \dashv C^a$ . Second, we have  $(f \circ \alpha_{a_1})(0) = f(0_1) = 0_2 = \alpha_{f(a_1)}(0)$  and  $(f \circ \alpha_{a_1})(1) = f(a_1) = \alpha_{f(a_1)}(1)$ . Third, we have  $(C^{a_1} \circ g)(a_2) = 1 \Leftrightarrow a_1 < g(a_1) \Leftrightarrow f(a_1) < a_2 \Leftrightarrow C^{f(a_1)}(a_2) = 1$ .

(2) First, if  $\chi = 0$  then  $x < \alpha^a(0) = a \Leftrightarrow C_a(x) = 0 \Leftrightarrow C_a(x) < 0$  whereas if  $\chi = 1$  then  $x < 1 = \alpha^a(x)$  and  $C_a(x) < 1$ . Hence  $C_a \dashv \alpha^a$ . Second, we have  $(g \circ \alpha^{a_2})(0) = g(a_2) = \alpha^{g(a_2)}(0)$  and  $(g \circ \alpha^{a_2})(1) = g(1_2) = 1_1 = \alpha^{g(a_2)}(1)$ . Third, we have  $(C_{a_2} \circ f)(a_1) = 0 \Leftrightarrow f(a_1) < a_2 \Leftrightarrow a_1 < g(a_2) \Leftrightarrow C_{g(a_2)}(a_1) = 0$ .

(3) Let  $x \in L$ . Then  $(i_a \circ \pi_a)(x) = i_a(x \wedge a) = x \wedge a < x$  and  $i_a \circ \pi_a < \text{id}$ . Let  $x \in [0, a]$ . Then  $x = x \wedge a = \pi_a(x) = (\pi_a \circ i_a)(x)$  and  $\pi_a \circ i_a = \text{id}$ . Let  $x \in L$ . Then  $x < [x(x < a) : 1(x \not< a)] = [\hat{i}_a(x)(x < a); \hat{i}_a(a)(x \not< a)] = (\hat{i}_a \circ \hat{\pi}_a)(x)$  and  $\text{id} < \hat{i}_a \circ \hat{\pi}_a$ . Let  $x \in [0, a]$ . Then  $(\hat{\pi}_a \circ \hat{i}_a)(x) = [\hat{\pi}_a(x)(x \neq a); \hat{\pi}_a(1)(x = a)] = x$  and  $\hat{\pi}_a \circ \hat{i}_a = \text{id}$ .

### Proof of proposition 4.2:

(1) Let  $f$  be balanced. Then  $g(a_2) = 1_1 \Leftrightarrow 1_1 < g(a_2) \Leftrightarrow 1_2 = f(1_1) < a_2 \Leftrightarrow a_2 = 1$ , and  $g$  is dense. Let  $g$  be dense. Then  $1_1 < (g \circ f)(1_1) \Rightarrow f(1_1) = 1_2$ , and  $f$  is balanced. Let  $f$  be dense. Then  $(f \circ g)(0_2) < 0_2 \Rightarrow g(0_2) = 0_1$ , and  $g$  is balanced.

Let  $g$  be balanced. Then  $f(a_1) = 0_2 \Leftrightarrow f(a_1) < 0_2 \Leftrightarrow a_1 < g(0_2) = 0_1 \Leftrightarrow a_1 = 0_1$ , and  $f$  is dense.

(2) Let  $f$  be epic. Then  $C_{a_2} \circ f = C_{g(a_2)} = C_{(g \circ f \circ g)(a_2)} = C_{(f \circ g)(a_2)} \circ f$ , so that  $C_{a_2} = C_{(f \circ g)(a_2)}$  and  $a_2 = (f \circ g)(a_2)$ . Let  $f$  be surjective. Then with  $a_2 = f(a_1)$  we have  $(f \circ g)(a_2) = (f \circ g \circ f)(a_1) = f(a_1) = a_2$  and  $f \circ g = \text{id}_2$ . Let  $f \circ g = \text{id}_2$ . Then  $g(a_2) = g(b_2) \Rightarrow a_2 = \text{id}_2(a_2) = (f \circ g)(a_2) = (f \circ g)(b_2) = \text{id}_2(b_2) = b_2$ . Let  $g$  be injective. Then  $g \circ g_1 = g \circ g_2 \Rightarrow (g \circ g_1)(a) = (g \circ g_2)(a) \Rightarrow g_1(a) = g_2(a)$ . Let  $g$  be monic. Then  $f_1 \circ f = f_2 \circ f \Rightarrow g \circ g_1 = (f_1 \circ f)^* = (f_2 \circ f)^* = g \circ g_1$ , so that  $g_1 = g_2 \Rightarrow f_1 = g_{1*} = g_{2*} = f_2$ .

(3) Let  $f$  be a monic. Then  $f(a_1) = f(b_1) \Rightarrow f \circ \alpha_{a_1} = \alpha_{f(a_1)} = \alpha_{f(b_1)} = f \circ \alpha_{b_1}$ , so that  $\alpha_{a_1} = \alpha_{b_1}$  and  $a_1 = b_1$ . Let  $f$  be injective. Then  $(f \circ g \circ f)(a_1) = f(a_1)$  and  $(g \circ f)(a_1) = a_1$ . Let  $g \circ f = \text{id}_1$ . Then  $a_1 = \text{id}_1(a_1) = (g \circ f)(a_1)$ . Let  $g$  be surjective. Then  $g_1 \circ g = g_2 \circ g \Rightarrow g_1(a_1) = (g_1 \circ g)(a_2) = (g_2 \circ g)(a_2) = g_2(a_1)$  for  $a_1 = g(a_2)$ . Let  $g$  be epic. Then  $f \circ f_1 = f \circ f_2 \Rightarrow g_1 \circ g = (f \circ f_1)^* = (f \circ f_2)^* = g_2 \circ g$ , so that  $g_1 = g_2 \Rightarrow f_1 = g_{1*} = g_{2*} = f_2$ .

#### Proof of proposition 4.3:

(1) If  $b < \Pi_\beta((a_\alpha))$  then  $\alpha = \beta \Rightarrow (i_\beta(b))_\alpha = b < a_\alpha$  and  $\alpha \neq \beta \Rightarrow (i_\beta(b))_\alpha = 0 < a_\alpha$ . If  $i_\beta(b) < (a_\alpha)$  then  $b = (i_\beta(b))_\beta < a_\beta = \Pi_\beta((a_\alpha))$ . Hence  $i_\beta \dashv \Pi_\beta$ . If  $(a_\alpha) < j_\beta(b)$  then  $\Pi_\beta((a_\alpha)) = a_\beta < (j_\beta(b))_\beta = b$ . If  $\Pi_\beta((a_\alpha)) < b$  then  $\alpha = \beta \Rightarrow a_\alpha < b = (j_\beta(b))_\alpha$  and  $\alpha \neq \beta \Rightarrow a_\alpha < 1 = (j_\beta(b))_\alpha$ . Hence  $\Pi_\beta \dashv j_\beta$ . If  $x < I_\beta(b) = b$  then  $x \in L_\beta$  and  $\sigma_\beta(x) = x < b$ . If  $\sigma_\beta < b$  then  $x \in L_\beta \Rightarrow x = \sigma_\beta(x) < b = I_\beta(b)$  and  $x \notin L_\beta \Rightarrow 1 = \sigma_\beta(x) < b \Rightarrow x < 1 = b = I_\beta(b)$ . Hence  $\sigma_\beta \dashv I_\beta$ . If  $b < \rho_\beta(x)$  then  $x \in L_\beta \Rightarrow I_\beta(b) = b < \rho_\beta(x) = x$  and  $x \notin L_\beta \Rightarrow b < \rho_\beta(x) = 0 \Rightarrow I_\beta(b) = b = 0 < x$ . If  $b = I_\beta(b) < x$  then  $x \in L_\beta$  and  $b < x = \rho_\beta(x)$ . Hence  $I_\beta \dashv \rho_\beta$ .

(2) Let  $p_\beta : L \rightarrow L_\beta$  be maps. Now  $a \in \times_\alpha L_\alpha \Rightarrow a = (\Pi_\alpha(a))$ . Hence the unique map  $\theta : L \rightarrow \times_\alpha L_\alpha$  with  $p_\beta = \Pi_\beta \circ \theta$  is  $\theta(x) = ((\Pi_\alpha \circ \theta)(x)) = (p_\alpha(x))$ . Since the partial order, minimal and maximal elements, joins and meets are computed pointwise,  $\theta$  will be a morphism in BPos, JCLatt or MCLatt iff each  $p_\alpha$  is.

(3) First, let  $q_\beta : L_\beta \rightarrow L$  be morphisms. Then the unique map  $\varphi : \dot{\cup}_\alpha L_\alpha \rightarrow L$  such that  $q_\beta = \varphi \circ I_\beta$  is given by  $\varphi(b) = (\varphi \circ I_\beta)(b) = q_\beta(b)$  for  $b \in L_\beta$ . Note that there is no ambiguity for  $x = 0$  or  $x = 1$  since  $q_\beta(0) = 0$  and  $q_\beta(1) = 1$ . It remains to prove that  $\varphi$  is isotone. Let  $x < x^*$ . Then  $x$  and  $x^*$  belong to the same component, say  $L_\beta$ , and  $\varphi(x) = q_\beta(x) < q_\beta(x^*) = \varphi(x^*)$ . Second,  $\Pi_\beta$  is the product in MCLatt, so that  $\Pi_\beta$  is the coproduct in MCLatt<sup>op</sup> and  $i_\beta$  is the coproduct in JCLatt. Third,  $\Pi_\beta$  is the product in JCLatt, so that  $\Pi_\beta$  is the coproduct in JCLatt<sup>op</sup> and  $j_\beta$  is the coproduct in MCLatt.

#### Proof of proposition 4.4:

(1) First, for  $x_2 \neq \mathbf{1}$  we have  $(F_\alpha \circ G_\alpha)(x_2) = F_\alpha(\alpha^*(x_2)) = (\alpha \circ \alpha^*)(x_2) < x_2$ , and for  $x_2 = \mathbf{1}$  we have  $(F_\alpha \circ G_\alpha)(\mathbf{1}) = F_\alpha(\mathbf{1}) = \mathbf{1}$ . On the other hand, for  $x_1 < a_1$  we have  $x_1 < (\alpha^* \circ \alpha)(x_1) = G_\alpha(\alpha(x_1)) = (G_\alpha \circ F_\alpha)(x_1) = G_\alpha$ , and for  $x_1 \not< a_1$  we have  $x_1 < \mathbf{1} = G_\alpha(\mathbf{1}) = (G_\alpha \circ F_\alpha)$ . Second, we have that  $(F_{\alpha_2} \circ F_{\alpha_1})(x_1) = \mathbf{1}$  iff  $x_1 \not< (G_{\alpha_1} \circ G_{\alpha_2})(1_3) = G_{\alpha_1}(\alpha_2^*(1_3)) = G_{\alpha_1}(a_2) = \alpha_1^*(a_2)$  iff  $F_{\alpha_2 \circ \alpha_1}(x_1) = \mathbf{1}$ , and

for  $x_1 < \alpha^*(a_2)$  we have  $F_{\alpha_2 \circ \alpha_1}(x_1) = (\alpha_2 \circ \alpha_1)(x_1) = (F_{\alpha_2} \circ F_{\alpha_1})(x_1)$ . Third, for  $x_1 < F^*(1_2)$  we have  $F_{\alpha_F}(x_1) = \alpha_F(x_1) = F(x_1)$ , and for  $x_1 \not< F^*(1_2)$  we have  $F_{\alpha_F}(x_1) = \mathbf{1} = F(x_1)$  since  $x_1 \not< F^*(1_2) \Leftrightarrow F(x_1) \not< 1_2 \Leftrightarrow F(x_1) = \mathbf{1}$ .  
 (2) First, for  $x_2 \in L_2$  we have  $(\alpha_F \circ \beta_F)(x_2) = \alpha_F(F^*(x_2)) = (F \circ F^*)(x_2) < x_2$ , and for  $x_1 < F^*(1_2)$  we have  $x_1 < (F^* \circ F)(x_1) = \beta_F(F(x_1)) = (\beta_F \circ \alpha_F)(x_1)$ . Second,  $\alpha_{F_2 \circ F_1}(x_1) = (F_2 \circ F_1)(x_1) = \alpha_{F_2}(F_1(x_1)) = (\alpha_{F_2} \circ \alpha_{F_1})(x_1)$ . Third,  $F_\alpha^*(1_2) = G_\alpha(1_2) = \alpha^*(1_2) = a_1$ , and  $\alpha_{F_\alpha}(x_1) = F_\alpha(x_1) = \alpha(x_1)$  for  $x_1 < a_1$ .  
 (3) If  $x_2 \neq \mathbf{1}$  then  $g^u(x_2) = g(x_2) = g^p(x_2) = \alpha^*(x_2) = G_\alpha(x_2)$ , whereas if  $x_2 = \mathbf{1}$  then  $g^u(x_2) = \mathbf{1} = G_\alpha(x_2)$ . Hence  $g^u = G_\alpha$  so that  $F = F_\alpha$ . Finally, if  $x_2 \in L_2$  then  $g^p(x_2) = g(x_2) = g^u(x_2) = F^*(x_2) = \beta_F(x_2)$ , so that  $g^p = \beta_F$  and  $\alpha = \alpha_F$ .

### Proof of proposition 5.1:

(1) Suppose that  $f(\Sigma_{L_1}) \subseteq \Sigma_{L_2} \cup \{0_2\}$ , and  $p_1 \in \Sigma_{L_1}$ . Then either  $f(p_1) = 0_2$  or  $f(p_1)$  is an atom. In either case there exists  $p_2 \in \Sigma_{L_2}$  such that  $f(p_1) < p_2$  and so  $p_1 < g(p_2)$ . Suppose that for each  $p_1 \in \Sigma_{L_1}$  there exists  $p_2 \in \Sigma_{L_2}$  such that  $p_1 < g(p_2)$ . Then  $f(p_1) < p_2$ , so that either  $f(p_1) = p_2$  or  $f(p_1) = 0_2$ .  
 (2) Let  $\alpha(T_1 A_1 \setminus K_1) \subseteq T_2 \alpha(A_1 \setminus K_1)$ , and for  $T_2 A_2 = A_2$  set  $A_1 = K_1 \cup \alpha^{-1}(A_2)$ . Then  $T_1 A_1 \subseteq K_1 \cup \alpha^{-1}(\alpha(T_1 A_1) \setminus K_1) \subseteq K_1 \cup \alpha^{-1}(T_2 \alpha(A_1 \setminus K_1)) \subseteq K_1 \cup f^{-1}(A_2) = A_2$  since  $A_1 \setminus K_1 = f^{-1}(A_2)$ . Let  $T_1(K_1 \cup \alpha^{-1}(A_2)) = K_1 \cup \alpha^{-1}(A_2)$  if  $T_2(A_2) = A_2$ , and set  $A_2 = T_2 \alpha(A_1 \setminus K_1)$ . Then  $\alpha(T_1 A_1 \setminus K_1) \subseteq \alpha(\alpha^{-1}(A_2)) \subseteq A_2 = T_2 \alpha(A_1 \setminus K_1)$  since  $A_1 \subseteq K_1 \cup \alpha^{-1}(\alpha(A_1 \setminus K_1)) \subseteq K_1 \cup \alpha^{-1}(T_2 \alpha(A_1 \setminus K_1)) = K_1 \cup \alpha^{-1}(A_2)$ .  
 (3) First,  $(f_2 \circ f_1)(\Sigma_{L_1}) \subseteq f_2(\Sigma_{L_2} \cup \{0_2\}) \subseteq \Sigma_{L_3} \cup \{0_3\}$ . Second, the kernel of a composition is given by  $K = K_1 \cup \alpha_1^{-1}(K_2)$ . Hence, if  $A_3$  is closed then so is  $K \cup (\alpha_2 \alpha_1)^{-1}(A_3) = (K_1 \cup \alpha_1^{-1}(K_2)) \cup \alpha_1^{-1}(\alpha_2^{-1}(A_3)) = K_1 \cup \alpha_1^{-1}(K_2 \cup \alpha_2^{-1}(A_3))$ . Third,  $K_{\alpha_2 \alpha_1} = (K_1 \cup \alpha_1^{-1}(K_2)) \cup (\alpha_2 \circ \alpha_1)^{-1}(K_3) = K_1 \cup \alpha_1^{-1}(K_2 \cup \alpha_2^{-1}(K_3)) = K_{(h_2 \circ f)}$ .

### Proof of proposition 5.2:

(1)  $A = \{p \in \Sigma \mid p \in A\} \subseteq \{p \in \Sigma \mid p < \bigvee A\} = \{p \in \Sigma \mid p < \pi(A)\} = (i \circ \pi)(A)$  and  $(\pi \circ i)(a) = \bigvee i(a) = \bigvee \{p \in \Sigma \mid p < a\} = 0$ , so that  $\pi \dashv i$ . Further, the closure  $i \circ \pi$  is simple, since  $(i \circ \pi)(\emptyset) = \{p \in \Sigma \mid p < \bigvee \emptyset = 0\} = \emptyset$  and  $(i \circ \pi)(\{p\}) = \{q \in \Sigma \mid p < \bigvee \{p\} = p\} = \{p\}$ .  
 (2) Note that  $A \in \mathbf{P}(\Sigma)^{i \circ \pi}$  if and only if  $A = \{p \in \Sigma \mid p < a\}$ . Let  $(i_2 \pi_2)(A_2) = A_2$ . Then  $K_1 = \{p \in \Sigma_1 \mid f(p_1) = 0_2\} = \{p \in \Sigma_2 \mid f(p_1) = 0_2 \& f(p_1) < \bigvee A_2\}$  and  $\alpha_f^{-1}(A_2) = \{p_1 \in \Sigma_1 \mid \alpha_f(p_1) \in A_2\} = \{p_1 \in \Sigma_1 \mid f(p_1) \neq 0 \& f(p_1) < \bigvee A_2\}$ , so that  $K_1 \cup \alpha_f^{-1}(A_2) = \{p_1 \in \Sigma_1 \mid f(p_1) < \bigvee A_2\} = \{p_1 \in \Sigma_1 \mid p_1 < f^*(\bigvee A_2)\}$  is closed.  
 (3) First, if  $f_\alpha(A_1) \subseteq A_2$  then  $\alpha(A_1 \setminus K_1) \subseteq T_2 \alpha(A_1 \setminus K_1) = f_\alpha(A_1) \subseteq A_2$  so that  $A_1 \setminus K_1 \subseteq \alpha^{-1}(\alpha(A_1 \setminus K_1)) \subseteq f^{-1}(A_2)$  and  $A_1 \subseteq K_1 \cup (A_1 \setminus K_1) \subseteq K_1 \cup \alpha^{-1}(A_2) = g_\alpha(A_2)$ . Second, if  $A_1 \subseteq g_\alpha(A_2) = K_1 \cup \alpha^{-1}(A_2)$  then we have  $A_1 \setminus K_1 \subseteq \alpha^{-1}(A_2)$  so that  $f_\alpha(A_1) = T_2 \alpha(A_1 \setminus K_1) \subseteq T_2 \alpha(\alpha^{-1}(A_2)) \subseteq T_2(A_2) = A_2$ . Third, let  $p_1 \in \Sigma_1$ . Then  $p_1 \notin K_1 \Rightarrow f_\alpha(\{p_1\}) = T_2 \alpha(\{p_1\} \setminus K_1) = T_2 f(\{p_1\}) = T_2 \{f(p_1)\} = \{f(p_1)\}$ , whereas  $p_1 \in K_1 \Rightarrow f_\alpha(\{p_1\}) = T_2 \alpha(\{p_1\} \setminus K_1) = T_2 f(\emptyset) = T_2(\emptyset) = \emptyset$ .  
 (4)  $\mathcal{L} : \alpha \mapsto f_\alpha$  and  $\mathcal{C} : f \mapsto \alpha_f$  are functors, since  $K_{\alpha_2 \alpha_1} = K_{\alpha_1} \cup \alpha_1^{-1}(K_{\alpha_2})$  so that  $g_{\alpha_2 \alpha_1}(A_3) = K_{\alpha_2 \alpha_1} \cup (\alpha_2 \circ \alpha_1)^{-1}(A_3) = K_{\alpha_1} \cup \alpha_1^{-1}(K_{\alpha_2} \cup \alpha_2^{-1}(A_3)) = (g_{\alpha_1} \circ g_{\alpha_2})(A_3)$ ,

and  $K_{\alpha_{f_2 \circ f_1}} = \{p_1 \in \Sigma_1 \mid (f_2 \circ f_1) = 0_3\} = K_{\alpha_{f_1}} \cup \alpha_1^{-1}(K_{\alpha_{f_2}}) = K_{\alpha_{f_2 \circ \alpha_{f_1}}}$  so that  $\alpha_{f_2 \circ f_1} = \alpha_{f_2} \circ \alpha_{f_1}$ . Let  $\varphi_\Sigma : \Sigma \rightarrow \mathcal{CL}\Sigma : p \mapsto \{p\}$  and  $\psi_L : L \rightarrow \mathcal{CL}L : a \mapsto \{p \in \Sigma_L \mid p < a\}$ . Then  $\varphi, \psi$  are natural,  $(\mathcal{CL}\alpha \circ \varphi_{\Sigma_1})(p) = \alpha_{f_\alpha}(\{p\}) = f_\alpha(\{p\}) = \{\alpha(p)\} = (\varphi_{\Sigma_2} \circ \alpha)(p)$  and  $(\mathcal{CL}f \circ \psi_{L_1})(a) = f_{\alpha_f}(\psi_{L_1}(a)) = \{p \in \Sigma_{L_1} \mid p < f(a)\} = (\psi_{L_2} \circ f)(a)$ , and form an equivalence,  $\mathcal{L}\varphi_\Sigma(A) = f_{\varphi_\Sigma}(A) = \varphi_\Sigma(TA) = \varphi_\Sigma(A) = \{\{p\} \mid \{p\} \subseteq A\} = \psi_{\mathcal{L}\Sigma}(A)$ .

### Proof of proposition 5.3 :

(1) First, the maps  $S_\alpha$  and  $P_\alpha$  are isotone, since for  $A_2 \subseteq B_2$  and  $A_1 \subseteq B_1$  we have  $x_1 \in S_\alpha(A_1) \Rightarrow (\exists x_2 \in A_2) \alpha(x_1) = x_2 \Rightarrow (\exists x_2 \in B_2) \alpha(x_1) = x_2 \Rightarrow x_1 \in S_\alpha(B_2)$  and  $x_2 \in P_\alpha(A_2) \Rightarrow (\exists x_1 \in A_1) \alpha(x_1) = x_2 \Rightarrow (\exists x_1 \in B_1) \alpha(x_1) = x_2 \Rightarrow x_2 \in P_\alpha(B_1)$ . Second,  $x_1 \in (S_\alpha \circ P_\alpha)(A_1) \Leftrightarrow \alpha(x_1) \in P_\alpha(A_1) \Leftrightarrow (\exists y_1 \in A_1) \alpha(x_1) = \alpha(y_1)$ . In particular,  $A_1 \subseteq (S_\alpha \circ P_\alpha)(A_1)$ , and if  $S_\alpha \circ P_\alpha = \text{id}$  or  $\alpha$  is injective we have that  $\alpha(x_1) = \alpha(\bar{x}_1) \Leftrightarrow (\exists y_1 \in \{\bar{x}_1\}) \alpha(x_1) = \alpha(y_1) \Leftrightarrow x_1 \in (S_\alpha \circ P_\alpha)(\{\bar{x}_1\}) = \{\bar{x}_1\} \Leftrightarrow x_1 = \bar{x}_1$  and  $x_1 \in (S_\alpha \circ P_\alpha)(A_1) \Leftrightarrow (\exists y_1 \in A_1) [\alpha(x_1) = \alpha(y_1) \Leftrightarrow x_1 = y_1] \Leftrightarrow x_1 \in A_1$ . Third,  $x_2 \in (P_\alpha \circ S_\alpha)(A_2) \Leftrightarrow (\exists x_1 \in S_\alpha(A_2)) \alpha(x_1) = x_2 \Leftrightarrow (\exists x_1 \in \Sigma_1) x_2 = \alpha(x_1) \in A_2$ . In particular,  $(P_\alpha \circ S_\alpha)(A_2) \subseteq A_2$ , and if  $P_\alpha \circ S_\alpha = \text{id}$  or  $\alpha$  is surjective we obtain  $x_2 \in \Sigma_2 \Leftrightarrow x_2 \in \{x_2\} = (P_\alpha \circ S_\alpha)(\{x_2\}) \Leftrightarrow (\exists x_1 \in \Sigma_1) x_2 = \alpha(x_1)$  and  $x_2 \in (P_\alpha \circ S_\alpha)(A_2) \Leftrightarrow (\exists x_1 \in \Sigma_1) x_2 = \alpha(x_1) \in A_2 \Leftrightarrow x_2 \in A_2$ .

(2) First,  $p = p \wedge 1 = p \wedge (a \vee a') = (p \wedge a) \vee (p \wedge a')$ , so that either  $p \wedge a = p \Leftrightarrow p < a$  or  $p \wedge a' = p \Leftrightarrow p < a'$ . In particular  $p \not< q'$  if and only if  $p = q$ , so that for  $A \subseteq \Sigma_B$  we obtain  $p < \bigvee A \Leftrightarrow p \not< (\bigvee A)' \Leftrightarrow \bigwedge A' \Leftrightarrow (\exists q \in A) p \not< q' \Leftrightarrow p \in A$ . Hence  $(\rho \circ \mu)(a) = \bigvee \{p \in \Sigma_B \mid p < a\} = a$  and  $(\mu \circ \rho)(A) = \{p \in \Sigma_B \mid p < \bigvee A\} = A$ . Second, let  $f(\Sigma_{B_1}) \subseteq \Sigma_{B_2}$ . Then  $g(0_2) = 0_1$ , since if  $p_1 < g(0_2)$  then  $f(p_1) < 0_2$  which is impossible by hypothesis, so that  $g(a_2) \wedge g(a'_2) = g(a_2 \wedge a'_2) = g(0_2) = 0_1$ . Further, for each  $p_1 \in \Sigma_{B_1}$  either  $f(p_1) < a_2$  and  $p_1 < g(a_1) < g(a_1) \vee g(a'_1)$  or  $f(p_1) < a'_2$  and  $p_1 < g(a'_1) < g(a_2) \vee g(a'_2)$ , so that  $g(a_2) \vee g(a'_2) = 1_1$ . Hence  $g(a'_2) = g(a_2)'$ . Third, let  $g(a'_2) = g(a_2)'$ . Then  $f(p_1) \neq 0_2$  since we have that  $f(a_1) < 0_2 \Leftrightarrow a_1 < g(0_2) = g(1'_2) = g(1_2)' = 1'_1 = 0_1$ . Further, if  $a_2 < f(p_1)$  then either  $p_1 < g(a_2)$ , so that  $a_2 < f(p_1) < a_2$  and  $a_2 = f(p_1)$ , or  $p_1 < g(a_2)' = g(a'_2)$ , so that  $a_2 < f(p_1) < a'_2$  and  $a_2 = 0_2$ . Hence  $f(p_1)$  is an atom.

### Proof of proposition 6.1 :

(1) The maps are isotone: if  $A \subseteq B$  then  $J(A) = \bigvee A < \bigvee B = J(B)$ , and if  $a < b$  then  $\ell(a) = [0, a] \subseteq [0, b] = \ell(b)$ . Further,  $(J \circ \ell)(a) = J([0, a]) = \bigvee [0, a] = a$ , and  $A \subseteq [0, \bigvee A] = \ell(\bigvee A) = (\ell \circ J)(a)$ .

(2) Let  $f \succ \theta$ . Then  $f = f \circ \text{id}_1 = f \circ J_1 \circ \ell_1 = J_2 \circ \theta \circ \ell_1$ . Further, let  $g = J_1 \circ \varphi \circ \ell_2$  where  $\theta \dashv \varphi$ . Then  $f \circ g = f \circ J_1 \circ \varphi \circ \ell_2 = J_2 \circ \theta \circ \varphi \circ \ell_2 < J_2 \circ \ell_2 = \text{id}_2$  and  $\text{id}_1 = J_1 \circ \ell_1 < J_1 \circ \varphi \circ \theta \circ \ell_1 < J_1 \circ \varphi \circ \ell_2 \circ J_2 \circ \theta \circ \ell_1 = g \circ f$ , so that  $f \dashv g$ .

(3) Suppose that  $\theta$  is strongly isotone, and define  $f : L_1 \rightarrow L_2 : a_1 \mapsto \bigvee \theta(\{a_1\})$ . Then we have  $f(\bigvee A_1) = \bigvee \theta(\{\bigvee A_1\}) = \bigvee \theta(A_1)$ , since  $\bigvee \{\bigvee A_1\} = \bigvee A_1$  so that  $\{\bigvee A_1\} \equiv A_1$ . Suppose that  $f \succ \theta$  and  $A_1 \ll B_1$ . Then  $\bigvee A_1 < \bigvee B_1$ , so that  $\bigvee \theta(A_1) = f(\bigvee A_1) < f(\bigvee B_1) = \bigvee \theta(B_1)$  and  $\theta(A_1) \ll \theta(B_1)$ .

(4) If  $f_1 \succ \theta_1$ ,  $f_2 \succ \theta_2$  then  $f_2 \circ f_1 \circ J_1 = f_2 \circ J_2 \circ \theta_1 = J_3 \circ \theta_2 \circ \theta_1$ , and  $f_2 \circ f_1 \succ \theta_2 \circ \theta_1$ . If  $f_\alpha \succ \theta_\alpha$  then  $(\bigvee_\alpha f_\alpha) \circ J_1 = \bigvee_\alpha (f_\alpha \circ J_1) = \bigvee_\alpha (J_2 \circ \theta_\alpha) = J_2 \circ (\bigcup_\alpha \theta_\alpha)$ , and  $\bigvee_\alpha f_\alpha \succ \bigcup_\alpha \theta_\alpha$ .

### Proof of proposition 6.2:

(1) First,  $f \in J(\mathbf{2}, L)$  iff  $f = f_a : [0 \mapsto 0; 1 \mapsto a]$ , whereas  $\theta \in J(P_0(\mathbf{2}), P_0(L))$  iff  $\theta = \theta_A : [\emptyset \mapsto \emptyset; \{1\} \mapsto A]$ . Second,  $P_{f_a} = \theta_{\{a\}}$  so that  $\bigcup_{a \in A} P_{f_a} = \theta_A$ . Third,  $f_a \succ \theta_A \Leftrightarrow f_a(\bigvee X) = \bigvee \theta_A(X) \Leftrightarrow a = f(1) = f(\bigvee \{1\}) = \bigvee \theta(\{1\}) = \bigvee A$ . Hence  $PS(\mathbf{2}, L) = L$  whereas  $BS(\mathbf{2}, L) = TS(\mathbf{2}, L) = FS(\mathbf{2}, L) = P_0(L)$ .

(2) First,  $f \in J(L, \mathbf{2})$  iff  $f = f_a : [x \mapsto 0 (x < a); 1 (x \not< a)]$ , whereas we have that  $\theta \in J(P_0(L), P_0(\mathbf{2}))$  iff  $\theta = \theta_A : [X \mapsto \emptyset (X \subseteq A); \{1\} (X \not\subseteq A)]$ . Second,  $P_{f_a}(\{x\}) = [\emptyset (x < a); \{1\} (x \not< a)]$  and so  $P_{f_a} = \theta_{[0, a]}$ . Third,  $f_a \succ \theta_A$  iff  $\bigvee X < a \Leftrightarrow f_a(\bigvee X) = 0 \Leftrightarrow \bigvee \theta_A(X) = 0 \Leftrightarrow X \subseteq A$ , iff  $A = [0, a]$  and  $\theta_A = P_{f_a}$ . Hence  $PS(L, \mathbf{2}) = BS(L, \mathbf{2}) = TS(L, \mathbf{2}) = L$  whereas  $FS(L, \mathbf{2}) = P_0(L)$ .

(3) We must give an example for which the inclusion  $\underline{\mathbf{BStruct}} \hookrightarrow \underline{\mathbf{TStruct}}$  is strict. Let  $\theta : P_0(L) \rightarrow P_0(L) : A \mapsto [A (1 \notin A); \{a\} \cup A (1 \in A)]$ . Then  $\theta$  preserves unions,  $\theta(\bigcup_\alpha A_\alpha) = \{a\} \cup A \Leftrightarrow 1 \in \bigcup_\alpha A_\alpha \Leftrightarrow (\exists \alpha) [1 \in A_\alpha \Leftrightarrow \theta(A_\alpha) = \{a\} \cup A] \Leftrightarrow \bigcup_\alpha \theta(A_\alpha) = \{a\} \cup A$ , and  $\text{id}_L \succ \theta$ , since if  $1 \notin A$  we have  $\text{id}_L(\bigvee A) = \bigvee A = \bigvee \theta(A)$  whereas if  $1 \in A$  we have  $\text{id}_L(\bigvee A) = \bigvee A = 1 = a \vee 1 = a \vee (\bigvee A) = \bigvee (\{a\} \cup A) = \bigvee \theta(A)$ . However, if there exists  $1 \neq b \not< a$  then we cannot express  $\theta$  as a union of power maps. Indeed, suppose that  $\theta = \bigcup_\alpha P_{f_\alpha}$  so that  $\theta(\{x\}) = \bigcup_\alpha P_{f_\alpha}(\{x\}) = \{f_\alpha(x)\}$ . Then  $\{f_\alpha(x)\} = \theta(\{x\}) = \{x\}$  for  $x \neq 1$  whereas  $\{f_\alpha(1)\} = \theta(\{1\}) = \{a, 1\}$ . Hence we would have  $\theta = P_{f_1} \cup P_{f_a}$  where  $f_1 : x \mapsto x$  and  $f_a : x \mapsto x (x \neq 1); a (x = 1)$ , the latter not being isotone since  $b < 1$  whereas  $f_a(b) = b \not< a = f_a(1)$ .

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